Bernoulli numbers and multiple zeta values

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Abstract: We show an apparently new expression of Bernoulli numbers, simultaneously we give an expression of multiple zeta values $\zeta(2m, 2m, \dots, 2m)$.

Key words: Bernoulli numbers; multiple zeta values.

1. Introduction. Bernoulli numbers B_n $(n = 1, 2, 3, \dots)$ are defined by

$$\frac{t}{e^t - 1} = \sum_{n=1}^{\infty} B_n \frac{t^n}{n!} \qquad (|t| < 2\pi). \tag{1.1}$$

In Gould [1], there are many formulas about those numbers. For example,

$$B_n = \sum_{k=0}^{n} \frac{1}{k+1} \sum_{j=0}^{k} (-1)^j \binom{k}{j} j^n,$$

and

$$B_n = \sum_{j=0}^{n} (-1)^j \binom{n+1}{j+1} \frac{n!}{(n+j)!} \sum_{k=0}^{j} (-1)^{j-k} \binom{j}{k} k^{n+j}.$$

Gould ended the paper [1] with the following conjecture.

> *The writer has seen no formula for B_n which does not require at least two actual summations. All the formulas we have quoted here are of this type.*

In this paper, we show an expression for B_n which needs only 'one' summation (Corollary 2.2). Simultaneously, we consider the Zagier multiple sum [2]

$$\zeta(m_1, m_2, \dots, m_n) := \sum_{k_1 > k_2 > \cdots > k_n > 0} \frac{1}{k_1^{m_1} k_2^{m_2} \cdots k_n^{m_n}}, \tag{1.2}$$

and give an explicit formula for $\zeta(2m, 2m, \dots, 2m)$ (Corollary 2.3).

2. Results. After I had completed my proof of the following result, Professor Masanobu Kaneko kindly informed me that the same result is included in an unpublished paper [3].

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Theorem 2.1. *Let l, m and n be positive integers, x be an indeterminate element. Then*

$$\frac{(-1)^{m+1} i}{(2\pi x)^m} {\sum}' \delta \exp\bigl((\pm\omega_{2m} \pm \omega_{2m}^2 \pm \cdots \pm \omega_{2m}^m)\pi i x\bigr) = 1 + \sum_{n=1}^{\infty} (-1)^n \zeta(\underbrace{2m, 2m, \dots, 2m}_{n}) x^{2mn}, \tag{2.1}$$

where

$$\omega_m^l = \exp(2\pi i l/m) \qquad (1 \le l \le m).$$

The symbol ${\sum}'$ means that all cases of $\pm$ are taken, namely it contains 2^m cases. And δ is defined by

$$\delta := \begin{cases} 1 & \text{if } -1 \text{ appears even times} \\ -1 & \text{if } -1 \text{ appears odd times} \end{cases}.$$

Proof. Using

$$\sin \pi\omega_{2m}^l x = \frac{\exp(i\pi\omega_{2m}^l x) - \exp(-i\pi\omega_{2m}^l x)}{2i}$$

and

$$\prod_{l=1}^{m} \omega_{2m}^l = \omega_{2m}^{m(m+1)/2} = i^{m+1},$$

we obtain

$$\prod_{l=1}^{m} \frac{\sin \pi\omega_{2m}^l x}{\pi\omega_{2m}^l x} = \frac{(-1)^{m+1} i}{(2\pi x)^m} \times {\sum}' \delta \exp\bigl((\pm\omega_{2m} \pm \omega_{2m}^2 \pm \cdots \pm \omega_{2m}^m)\pi i x\bigr).$$

On the other hand, using

$$\frac{\sin \pi t}{\pi t} = \prod_{n=1}^{\infty} \left(1 - \frac{t^2}{n^2}\right),$$

we have

$$\prod_{l=1}^{m}\frac{\sin\pi\omega_{2m}^{l}x}{\pi\omega_{2m}^{l}x}=\prod_{l=1}^{m}\prod_{n=1}^{\infty}\left(1-\frac{\omega_{m}^{l}x^{2}}{n^{2}}\right)$$

$$=\prod_{n=1}^{\infty}\left(1-\frac{x^{2m}}{n^{2m}}\right)$$

$$=\left(1-\frac{x^{2m}}{1^{2m}}\right)\left(1-\frac{x^{2m}}{2^{2m}}\right)\left(1-\frac{x^{2m}}{3^{2m}}\right)\cdots$$

$$=1-\left(\sum_{k=1}^{\infty}\frac{1}{k^{2m}}\right)x^{2m}+\left(\sum_{k_1>k_2>0}\frac{1}{k_1^{2m}k_2^{2m}}\right)x^{4m}$$

$$-\left(\sum_{k_1>k_2>k_3>0}\frac{1}{k_1^{2m}k_2^{2m}k_3^{2m}}\right)x^{6m}+\cdots$$

$$=1+\sum_{n=1}^{\infty}(-1)^{n}\zeta(\underbrace{2m,2m,\ldots,2m}_{n})x^{2mn}.$$

□

By (2.1) and

$$B_{2m}=\frac{(-1)^{m-1}2(2m)!}{(2\pi)^{2m}}\zeta(2m),$$

Corollary 2.2. *we have*

$$B_{2m}=\frac{-2i^{3m+1}(2m)!}{2^{3m}(3m)!}\times\sum{}'\,\delta(\pm\omega_{2m}\pm\omega_{2m}^{2}\pm\cdots\pm\omega_{2m}^{m})^{3m}. \tag{2.2}$$

This formula includes only 'one' summation.

Corollary 2.3. *Using* (2.1)*, we obtain*

$$\zeta(\underbrace{2m,2m,\ldots,2m}_{n})=\frac{(-1)^{(n+1)(m+1)}i^{m+1}\pi^{2nm}}{2^{m}(2nm+m)!}\times\sum{}'\,\delta(\pm\omega_{2m}\pm\omega_{2m}^{2}\pm\cdots\pm\omega_{2m}^{m})^{2nm+m}. \tag{2.3}$$

Example 2.4.

$$\zeta(\underbrace{2,2,\ldots,2}_{n})=\frac{(-1)^{2n+2}(-1)\pi^{2n}}{2(2n+1)!}\sum{}'\,\delta\big(\pm(-1)\big)^{2n+1}=\frac{\pi^{2n}}{(2n+1)!}. \tag{2.4}$$

$$\zeta(\underbrace{4,4,\ldots,4}_{n})=\frac{(-1)^{3n+3}(-i)\pi^{4n}}{4(4n+2)!}\sum{}'\,\delta\big(\pm i\pm(-1)\big)^{4n+2}=\frac{2^{2n+1}\pi^{4n}}{(4n+2)!}. \tag{2.5}$$

Remark. We can obtain (2.5) by another method. Note that

$$\frac{2}{\pi^2t^2}\sin(\frac{1+i}{2}\pi t)\sin(\frac{1-i}{2}\pi t)=\frac{\cosh\pi t-\cos\pi t}{\pi^2t^2}=\sum_{n=0}^{\infty}\frac{2\pi^{4n}t^{4n}}{(4n+2)!}.$$

On the other hand, we have

$$\frac{2}{\pi^2t^2}\sin(\frac{1+i}{2}\pi t)\sin(\frac{1-i}{2}\pi t)$$

$$=\prod_{n=1}^{\infty}\left(1-\frac{it^2}{2n^2}\right)\times\prod_{n=1}^{\infty}\left(1+\frac{it^2}{2n^2}\right)$$

$$=\prod_{n=1}^{\infty}\left(1+\frac{t^4}{4n^4}\right)$$

$$=\left(1+\frac{t^4}{4\cdot1^4}\right)\left(1+\frac{t^4}{4\cdot2^4}\right)\left(1+\frac{t^4}{4\cdot3^4}\right)\cdots$$

$$=1+\left(\sum_{k=1}^{\infty}\frac{4^{-1}}{k^4}\right)t^4+\left(\sum_{k_1>k_2>0}\frac{4^{-2}}{k_1^4k_2^4}\right)t^8$$

$$+\left(\sum_{k_1>k_2>k_3>0}\frac{4^{-3}}{k_1^4k_2^4k_3^4}\right)t^{12}+\cdots$$

$$=1+\sum_{n=1}^{\infty}4^{-n}\zeta(\underbrace{4,4,\ldots,4}_{n})t^{4n}.$$

Hence (2.5) follows.

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References

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