Continuity of Sobolev functions of variable exponent on metric spaces

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Abstract: Our aim in this paper is to discuss continuity of Sobolev functions of variable exponent on metric spaces in the limiting case of Sobolev's imbedding theorem.

Key words: Hölder continuity; differentiability; weighted Sobolev spaces; A_p-weight; p-Poincaré inequality; Sobolev's imbedding theorem of variable exponent.

1. Introduction. Sobolev functions are defined usually as functions whose weak derivatives (of first order) belong to some Lebesgue L^p class locally. Hence it is important to discuss the case that Sobolev functions are continuous and further have usual derivatives which are equal to weak derivatives. It is well known that this is true if p is greater than the space dimension n (see e.g., Stein [11, Chap. VIII]). There are known facts which treat the borderline case $p = n$, by the authors [5, 6, 7, 8] and [10]. Recently, Björn [1], Hajłasz-Koskela [2] and the authors [9] discussed continuity properties for Sobolev functions in the metric space setting; these results can be used to treat the differentiability of Sobolev functions defined in the Euclidean spaces. Our aim in the present note is to extend those results to Sobolev functions of variable exponent in the metric space setting. For this purpose we prepare some definitions.

Let X be a metric space with a metric d. For simplicity, we write $|x - y|$ instead of $d(x, y)$. We denote by $B(x, r)$ the open ball centered at $x \in X$ with radius $r > 0$. For $\sigma > 0$, we write

$$\sigma B(x, r) = B(x, \sigma r).$$

Let μ be a Borel measure on X. Assume that $\mu(B) < \infty$ and there exist constants $C > 0$ and $s \geqq 1$ such that

$$\frac{\mu(B')}{\mu(B)} \geqq C\left(\frac{r'}{r}\right)^s \tag{1.1}$$

for all balls $B = B(x, r)$ and $B' = B(x', r')$ with $x' \in B$ and $0 < r' \leqq r$. Note that μ is a doubling measure on X, that is, there exists a constant $C' > 0$ such that

$$\mu(B(x, 2r)) \leqq C'\mu(B(x, r)) \tag{1.2}$$

for all $x \in X$ and $r > 0$.

We say that a pair (u, g) of functions in $L^{p_0}_{loc}(X; \mu)$ satisfies a p_0-Poincaré inequality (on rings), $1 \leqq p_0 < \infty$, if for every c_1 and c_2 with $c_2 > c_1 > 1$ there are constants $M > 0$ and $\sigma \geqq 1$ such that

$$\fint_{A(r,r')} |u(y) - u_{A(r,r')}|\, d\mu \leqq Mr\left(\fint_{\sigma A(r,r')} |g|^{p_0}\, d\mu\right)^{1/p_0} \tag{1.3}$$

whenever $c_1 r' < r < c_2 r'$, where $A(r, r') = B(x, r) - B(x, r')$, $\sigma A(r, r') = B(x, \sigma r) - B(x, \sigma^{-1}r')$ and

$$u_G = \fint_G u\, d\mu = \frac{1}{\mu(G)}\int_G u\, d\mu$$

for Borel sets $G \subset X$. If (1.3) holds for $r' = 0$, then the pair (u, g) is said to satisfy the usual p_0-Poincaré inequality on balls. Under certain assumptions, the usual p_0-Poincaré inequality on balls implies our p_0-Poincaré inequality; see e.g. [2, Theorem 9.7].

When $X = \mathbf{R}^n$, Björn ([1, Theorem 1.5 and Theorem 3.1]) proved that if $p_0 > s$ and u is a function in the weighted Sobolev space $W^{1,p_0}(\mathbf{R}^n; \mu)$, then u can be modified on a set of μ-measure zero so that it is locally Hölder continuous in $\mathbf{R}^n$ and totally differentiable μ-a.e. in $\mathbf{R}^n$. In the previous paper [9], we extended her results by proving Hölder continuity of Sobolev functions when $p_0 \geqq s$, in the metric space setting.

In the present note, we consider a continuous function $p : X \to [s, \infty)$ (called the variable exponent on X), and discuss the log-Hölder continuity of

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weighted Sobolev functions of variable exponent, as an extention of the authors' result [9, Theorem 1]. When $X = \mathbf{R}^n$, we are concerned with differentiability of functions in the weighted Sobolev space $W^{1,p(\cdot)}(\mathbf{R}^n;\mu)$ of variable exponent, which is defined by

$$W^{1,p(\cdot)}(\mathbf{R}^n;\mu) = \{u \in L^{p(\cdot)}(\mathbf{R}^n;\mu) : |\nabla u| \in L^{p(\cdot)}(\mathbf{R}^n;\mu)\};$$

for fundamental properties of this space, see, for example, Kováčik and Rákosník [4].

2. Statement of results. Consider a positive function $\Phi_q(r)$ with the following properties:

(φ1) $\Phi_q(r)$ is of the form $r^q\varphi(r)$, where $1 \leqq q < \infty$ and φ is a positive nondecreasing function on $(0,\infty)$. The value $\Phi_q(0)$ is defined to be zero.

(φ2) There exists $c > 1$ such that

$$c^{-1}\varphi(r) \leqq \varphi(r^2) \leqq c\varphi(r)$$

whenever $r > 0$.

We see that $\Phi_q(\cdot)$ is continuous on $[0,\infty)$.

For a continuous function $p : X \to (1,\infty)$, we define $p_*(B) := \inf\{p(x) : x \in B\}$ and $p^*(B) := \sup\{p(x) : x \in B\}$. For simplicity, set $p_* = p_*(B_0)$ for fixed ball $B_0 = B(x_0,r_0)$. We consider the function

$$\kappa(r) = \left(\int_0^r [t^{-\varepsilon\omega(t)}\varphi(t^{-1})]^{-1/(p_*-1)}t^{-1}dt\right)^{(p_*-1)/p_*}$$

for $0 < \varepsilon < 1$, where $\omega(r) = \inf_{x\in A(r,r/2)} |p(x) - p_*|$. In this note we assume that

(ω) ω satisfies the doubling condition on $[0,\infty)$, that is, there exists $c > 1$ such that

$$c^{-1}\omega(r) \leqq \omega(2r) \leqq c\omega(r) \quad \text{for} \quad r \geqq 0.$$

Our main aim in this note is to show the following result, which is an extension of [2, Theorem 5.1] and [9, Theorem 1].

Theorem 2.1. *Let X be a connected metric space and let μ be a Borel measure on X satisfying the decay condition* (1.1) *with $s = p_*$. Assume further that a pair (u,g) satisfies the p_*-Poincaré inequality in X, $\kappa(1) < \infty$ and*

$$\int_X \Phi_{p(x)}(|g(x)|)\,d\mu(x) < \infty. \tag{2.1}$$

Then u can be modified on a set of μ-measure zero so that it is locally κ-Hölder continuous on B_0. Moreover, u satisfies

$$|u(x) - u(y)| \leqq Mr^{1-\varepsilon} + Mr_0\,\kappa(r)\left(\fint_{c\sigma B_0} \Phi_{p(z)}(|g(z)|)\,d\mu(z)\right)^{1/p_*}$$

for all x, $y \in B = B(x_0,r)$ with $0 < r < r_0$, where c is a positive constant, $0 < \varepsilon < 1$ and M is a positive constant depending on ε.

This can be proved in a way similar to the proof of [9, Theorem 1] with some needed modifications, and hence we omit the proof.

Remark 2.2. Assume that $p(\cdot)$ is a function on X defined by

$$p(x) = s + \frac{a\log[\log(1/r)]}{\log(1/r)} + \frac{a'}{\log(1/r)},$$

for $r \leq r_0 < 1/e$, where $r = |x - x_0|$, $a > 0$ and a' is a real number; set $p(x) = p(x_0 + r_0(x - x_0)/|x - x_0|)$ when $|x - x_0| > r_0$. Further, consider

$$\varphi(r) = [\log(1 + r)]^b$$

for a real number b. If $b > s - a - 1$, then, taking ε such that $(s-b-1)/a < \varepsilon < 1$, we see that $\kappa(1) < \infty$ and $\kappa(r) \leqq M[\log(1/r)]^{-(\varepsilon a+b+1-s)/s}$ for small $r > 0$.

The simplest case is as follows:

Corollary 2.3. *Let $p(\cdot)$ be as above. If $u \in W^{1,p(\cdot)}(B_0;\mu)$, then u can be modified on a set of measure zero to satisfy*

$$|u(x) - u(y)| \leqq M[\log(1/|x - y|)]^{-A}$$

for all x, $y \in B(x_0,r)$ with $r = \min\{r_0/2, 1/4\}$, when $0 < A < (a + 1 - s)/s$.

The log-Hölder continuity does not always hold when $a < s - 1$, as will be shown in the last section.

A positive measurable function w on $\mathbf{R}^n$ is called an A_q weight (written as $w \in (A_q)$) if there exists a positive constant C_q such that

$$\fint_B w(x)\,dx \left(\fint_B w(x)^{1/(1-q)}\,dx\right)^{q-1} \leqq C_q$$

for all balls B, where $1 < q < \infty$; we say that w is an A_1 weight if there exists a positive constant C_1 such that

$$\fint_B w(x)\,dx \leqq C_1 \operatorname{ess\,inf}_B w$$

for all balls B. Note that if w is an A_q weight, then $d\mu = wdx$ satisfies the decay condition (1.1) with $s = nq$, on account of [3, Section 15.5].

In view of [3, Section 15.26], we can show that the p_*-Poincaré inequality is valid for $(u, |\nabla u|)$ and

$d\mu = wdx$ with $u \in W^{1,p}(\mathbf{R}^n;\mu)$ and $w \in (A_{p_*})$. Hence we have the following result as an extension of Björn [1, Theorem 1.5], the first author [5, 6, 7] and the authors [8, 9, 10].

Corollary 2.4. *Let $1 < p_* < \infty$, $w \in (A_{p_*})$ and $\kappa(1) < \infty$. Assume further that μ satisfies* (1.1) *with $s = p_*$, where $d\mu = wdx$. Let u be a function in $W^{1,p(\cdot)}(\mathbf{R}^n;\mu)$ satisfying*

$$\int_{\mathbf{R}^n} \Phi_{p(x)}(|\nabla u(x)|)\, d\mu(x) < \infty. \tag{2.2}$$

Then u can be modified on a set of measure zero so that it becomes a locally κ-Hölder continuous function on $\mathbf{R}^n$ satisfying

$$|u(x) - u(y)| \leqq Mr^{1-\varepsilon} + Mr_0\ \kappa(r)\left(⨍_{cB_0} \Phi_{p(z)}(|\nabla u(z)|)\, d\mu(z)\right)^{1/p}$$

for all $x, y \in B = B(x_0, r)$ with $0 < r < r_0$, where c is a positive constant, $0 < \varepsilon < 1$ and M is a positive constant depending on ε.

Corollary 2.4 can be proved in the same way as Corollary 1 in [9].

We say that a function u on $\mathbf{R}^n$ is totally differentiable at x_0 if

$$\lim_{x \to x_0} \frac{|u(x) - u(x_0) - a \cdot (x - x_0)|}{|x - x_0|} = 0$$

for some $a \in \mathbf{R}^n$. By using Corollary 2.4, we can prove the differentiability of Sobolev functions in the same way as Theorem 2 in [9]; see also the book by Stein [11].

Theorem 2.5. *Let $1 < p_* < \infty$, $w \in (A_{p_*})$ and $\kappa(1) < \infty$. Assume further that μ satisfies* (1.1) *with $s = p_*$, where $d\mu = wdx$. Let u be a function in $W^{1,p(\cdot)}(\mathbf{R}^n;\mu)$ satisfying* (2.2). *Then u can be modified on a set of measure zero so that it becomes totally differentiable a.e. on $\mathbf{R}^n$.*

3. Sharpness. Let $\omega(r)$ be a nonnegative increasing continuous function on the interval $[0, \infty)$ such that $\omega(r) \leqq a \log(\log(1/r))/\log 1/r$ for small $r > 0$. Consider

$$p(y) = n + \omega(|y|)$$

and

$$\varphi(t) = [\log(1 + t)]^b.$$

In this case, $p_* = n$. Further, if $a + b < n - 1$, then $\kappa(1) = \infty$, and we can find a function u satisfying

(i) $\displaystyle\int_{B_0} \Phi_{p(x)}(|\nabla u(x)|)dx < \infty$;

(ii) $\displaystyle\lim_{x \to 0} u(x) = \infty$.

This implies that Theorem 2.1 is best possible as to the Hölder exponent. In fact, by considering $b = 0$, Corollary 2.3 is seen to be sharp as to the Hölder exponent.

To show this, consider the function

$$u(x) = [\log(1/|x|)]^\delta \quad \text{on} \quad B_0,$$

where $0 < \delta < 1$ and $B_0 = B(0, r_0)$ with $0 < r_0 < 1/e$. If $a + b < n(1 - \delta) - 1$, then we see that

$$|\nabla u(x)| \leqq M_1 |x|^{-1}[\log(1/|x|)]^{\delta - 1},$$

so that

$$\begin{aligned} \int_{B_0} \Phi_{p(x)}(|\nabla u(x)|)dx &\leqq M_2 \int_{B_0} |x|^{-(n+\omega(|x|))} \\ &\quad \times [\log(1/|x|)]^{(\delta-1)(n+\omega(|x|))}\varphi(|x|^{-1})dx \\ &\leqq M_3 \int_0^{r_0} (\log 1/t)^{(\delta-1)n+a+b} t^{-1} dt < \infty, \end{aligned}$$

where M_1, M_2, M_3 are positive constants.

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