A general optimal inequality for warped products in complex projective spaces and its applications

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Abstract: We prove a general optimal inequality for warped products in complex projective spaces and determine warped products which satisfy the equality case of the inequality. Two non-immersion theorems are obtained as immediate applications.

Key words: Warped product; inequality; complex projective space; non-immersion theorem.

1. Introduction. Let N_1 and N_2 be Riemannian manifolds of positive dimension n_1 and n_2, equipped with Riemannian metrics g_1 and g_2, respectively. Let f be a positive function on N_1. The warped product $N_1 \times_f N_2$ is defined to be the product manifold $N_1 \times N_2$ with the warped metric: $g = g_1 + f^2 g_2$ (see [7]).

For a warped product $N_1 \times_f N_2$, we denote by $\mathcal{D}_1$ the set of horizontal vector fields, i.e., vector fields on $N_1 \times_f N_2$ obtained from the horizonal lift of tangent vector fields of N_1; by $\mathcal{D}_2$ the set of vertical vector fields, i.e., vector fields obtained from the vertical lift of tangent vector fields of N_2. Denote by $\mathcal{H}$ and $\mathcal{V}$ the vector bundles over $N_1 \times_f N_2$ consisting of vectors tangent to leaves and to fibers, respectively.

Let $\phi : N_1 \times_f N_2 \to M$ be an isometric immersion of a warped product into a Riemannian manifold. Denote by h the second fundamental form of ϕ. Let $\operatorname{tr} h_1$ and $\operatorname{tr} h_2$ be the trace of h restricted to N_1 and N_2, respectively, i.e.,

$$\operatorname{tr} h_1 = \sum_{\alpha=1}^{n_1} h(e_\alpha, e_\alpha), \quad \operatorname{tr} h_2 = \sum_{t=n_1+1}^{n_1+n_2} h(e_t, e_t)$$

for orthonormal vector fields $e_1, \ldots, e_{n_1}$ in $\mathcal{H}$ and $e_{n_1+1}, \ldots, e_{n_1+n_2}$ in $\mathcal{V}$, respectively. The immersion ϕ is called mixed totally geodesic if $h(X, Z) = 0$ for any X in $\mathcal{H}$ and Z in $\mathcal{V}$.

A submanifold N of a Kaehler manifold (M, g, J) is called totally real if the complex structure J carries each tangent space of N into its corresponding normal space [4]. A totally real submanifold N in M with $\dim_{\mathbf{R}} N = \dim_{\mathbf{C}} M$ is known as a Lagrangian submanifold [1].

In [3], the author investigated warped products in complex hyperbolic spaces and obtain the following.

Theorem A. *Let $\phi : N_1 \times_f N_2 \to CH^m(4c)$ be an isometric immersion of a warped product into the complex hyperbolic m-space $CH^m(4c)$. Then we have*

$$\frac{\Delta f}{f} \leq \frac{(n_1+n_2)^2}{4n_2} H^2 + n_1 c, \tag{1.1}$$

where $n_i = \dim N_i$ $(i = 1, 2)$, H^2 is the squared mean curvature of ϕ, and Δ is the Laplacian of N_1.

The equality sign of (1.1) holds identically if and only if we have (1) ϕ is mixed totally geodesic, (2) $\operatorname{tr} h_1 = \operatorname{tr} h_2$ and (3) $J\mathcal{H} \perp \mathcal{V}$.

In [3] the author applied Theorem A to obtain some non-immersion theorems.

In this article, we study warped products in complex projective spaces and obtain the following.

Theorem 1. *Let $\phi : N_1 \times_f N_2 \to CP^m(4c)$ be an arbitrary isometric immersion of a warped product into the complex projective m-space $CP^m(4c)$ of constant holomorphic sectional curvature $4c$. Then we have*

$$\frac{\Delta f}{f} \leq \frac{(n_1+n_2)^2}{4n_2} H^2 + (3 + n_1)c. \tag{1.2}$$

The equality sign of (1.2) holds identically if and only if we have (1) $n_1 = n_2 = 1$, (2) f is an eigenfunction of the Laplacian of N_1 with eigenvalue $4c$, and (3) ϕ is totally geodesic and holomorphic.

As an immediate application, we obtain the following non-immersion theorem.

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Theorem 2. *If f is a positive function on a Riemannian n_1-manifold N_1 such that $(\Delta f)/f > 3 + n_1$ at some point $p \in N_1$, then, for any Riemannian manifold N_2, the warped product $N_1 \times_f N_2$ does not admit any isometric minimal immersion into $CP^m(4)$ for any m.*

For totally real minimal immersions, Theorem 2 can be sharpen as the following.

Theorem 3. *If f is a positive function on a Riemannian n_1-manifold N_1 such that $(\Delta f)/f > n_1$ at some point $p \in N_1$, then, for any Riemannian manifold N_2, the warped product $N_1 \times_f N_2$ does not admit any isometric totally real minimal immersion into $CP^m(4)$ for any m.*

In the last section, we provide examples to show that Theorems 1, 2 and 3 are sharp.

2. Preliminaries. Let N be an n-dimensional Riemannian manifold isometrically immersed in a Riemannian manifold M. We denote by $\langle\ ,\ \rangle$ the inner product for N as well as for M.

For any vector X tangent to N we put

$$JX = PX + FX,$$

where PX and FX are the tangential and the normal components of JX, respectively. Thus P is a well-defined endomorphism of the tangent bundle TN satisfying

$$\langle PX, Y\rangle = -\langle X, PY\rangle .$$

We denote by ∇ and $\tilde\nabla$ the Levi-Civita connections of N and M, respectively. Then the Gauss and Weingarten formulas are given respectively by

$$\tilde\nabla_X Y = \nabla_X Y + h(X,Y), \tag{2.1}$$

$$\tilde\nabla_X \xi = -A_\xi X + D_X\xi \tag{2.2}$$

for X, Y tangent to N and ξ normal to N, where h denotes the second fundamental form, D the normal connection and A the shape operator.

The mean curvature vector $\overrightarrow{H}$ is defined by $\overrightarrow{H} = (1/n)\operatorname{tr} h$. The squared mean curvature is given by $H^2 = \left\langle \overrightarrow{H}, \overrightarrow{H}\right\rangle$. A submanifold N is called minimal (respectively, totally geodesic) if its mean curvature vector (respectively, its second fundamental form) vanishes identically.

For the second fundamental form h, we define

$$\begin{aligned}(\bar\nabla_X\sigma)(Y,Z) &= D_X(\sigma(Y,Z)) \\ &\quad - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z).\end{aligned} \tag{2.3}$$

The equation of Codazzi is given by

$$\begin{aligned}&(\tilde R(X,Y)Z)^\perp \\ &\quad = (\bar\nabla_X\sigma)(Y,Z) - (\bar\nabla_Y\sigma)(X,Z)\end{aligned} \tag{2.4}$$

where $(\tilde R(X,Y)Z)^\perp$ is the normal component of $\tilde R(X,Y)Z$ and $\tilde R$ is the curvature tensors of M.

The scalar curvature of N is given by

$$\tau = \sum_{1\le i<j\le n} K(e_i \wedge e_j),$$

where $K(e_i \wedge e_j)$ is the sectional curvature of the plane section spanned by e_i and e_j.

For a differentiable function φ on N, the Laplacian of φ is defined by

$$\Delta\varphi = \sum_{j=1}^n \{(\nabla_{e_j} e_j)\varphi - e_j e_j \varphi\},$$

where $e_1, \dots, e_n$ is an orthonormal frame.

The Riemann curvature tensor $\tilde R$ of $CP^m(4c)$ is given by

$$\begin{aligned}&\tilde R(X,Y;Z,W) = c\{\langle X,W\rangle\langle Y,Z\rangle \\ &\quad - \langle X,Z\rangle\langle Y,W\rangle + \langle JX,W\rangle\langle JY,Z\rangle \\ &\quad - \langle JX,Z\rangle\langle JY,W\rangle + 2\langle X,JY\rangle\langle JZ,W\rangle\}.\end{aligned} \tag{2.5}$$

For a submanifold N of $CP^m(4c)$, the equation of Gauss is given by

$$\begin{aligned}&\langle R(X,Y)Z,W\rangle = \langle h(X,W), h(Y,Z)\rangle \\ &\quad - \langle h(X,Z), h(Y,W)\rangle + c\{\langle X,W\rangle\langle Y,Z\rangle \\ &\quad - \langle X,Z\rangle\langle Y,W\rangle + \langle JY,Z\rangle\langle JX,W\rangle \\ &\quad - \langle JX,Z\rangle\langle JY,W\rangle + 2\langle X,JY\rangle\langle JZ,W\rangle\},\end{aligned} \tag{2.6}$$

where R is the Riemannian curvature tensor of N. From (2.6) we know that the scalar curvature and the squared mean curvature of N satisfy

$$2\tau = n^2H^2 - ||h||^2 + n(n-1)c + 3c\,||P||^2, \tag{2.7}$$

where $||h||^2$ denotes the squared norm of the second fundamental form and

$$||P||^2 = \sum_{i,j=1}^n \langle e_i, Pe_j\rangle^2 \tag{2.8}$$

is the squared norm of the endomorphism P.

Let $N_1 \times_f N_2$ be a warped product. Then, for unit vector fields X, Y in $\mathcal{D}_1$ and Z in $\mathcal{D}_2$, we have

$$\begin{aligned}&\nabla_X Z = \nabla_Z X = (X \ln f)Z, \\ &\langle \nabla_X Y, Z\rangle = 0\end{aligned} \tag{2.9}$$

which implies that [7, page 210]

$$K(X \wedge Z) = \frac{1}{f}\{(\nabla_X X)f - X^2 f\}. \tag{2.10}$$

Thus, if $e_1, \dots, e_{n_1}$ are orthonormal horizontal vectors and z a unit vertical vector field, we have

$$\frac{\Delta f}{f} = \sum_{\alpha=1}^{n_1} K(e_\alpha \wedge z). \tag{2.11}$$

Let n be a natural number ≥ 2 and $n_1, \dots, n_k$ be k natural numbers. If $n_1 + \dots + n_k = n$, then $(n_1, \dots, n_k)$ is called a partition of n.

We recall the following general algebraic lemma from [2].

Lemma 1. *Let $a_1, \dots, a_n$ be n real numbers and let k be an integer in $[2, n-1]$. Then, for any partition $(n_1, \dots, n_k)$ of n, we have*

$$\begin{aligned} &\sum_{1\leq i_1<j_1\leq n_1} a_{i_1}a_{j_1} + \sum_{n_1+1\leq i_2<j_2\leq n_1+n_2} a_{i_2}a_{j_2} \\ &+ \dots + \sum_{n_1\cdots+n_{k-1}+1\leq i_k<j_k\leq n} a_{i_k}a_{j_k} \\ &\geq \frac{1}{2k}\Big\{(a_1+\dots+a_n)^2 - k(a_1^2+\dots+a_n^2)\Big\}, \end{aligned} \tag{2.12}$$

with the equality holding if and only if

$$\begin{aligned} &a_1 + \dots + a_{n_1} = \cdots \\ &= a_{n_1+\dots+n_{k-1}+1} + \dots + a_n. \end{aligned} \tag{2.13}$$

In this article, we use the following convention on the range of indices unless mentioned otherwise:

$$\begin{aligned} &j, k, \ell = 1, \dots, n_1 + n_2; \\ &\alpha, \beta = 1, \dots, n_1; \\ &s, t = n_1 + 1, \dots, n_1 + n_2. \end{aligned}$$

3. Proofs of Theorems 1, 2 and 3. Let $\phi : N_1 \times_f N_2 \to CP^m(4c)$ be an isometric immersion of a warped product $N_1 \times_f N_2$ into $CP^m(4c)$.

If we put

$$\eta = 2\tau - \frac{n^2}{2}H^2 - n(n-1)c - 3c||P||^2, \tag{3.1}$$

then (2.7) and (3.1) imply

$$n^2H^2 = 2\eta + 2||h||^2. \tag{3.2}$$

Let $e_1, \dots, e_{2m}$ be an orthonormal frame such that $e_1, \dots, e_{n_1}$ are in $\mathcal{H}$, $e_{n_1+1}, \dots, e_{n_1+n_2}$ are in $\mathcal{V}$, and e_{n+1} is in the direction of the mean curvature vector. Then (3.2) can be written as

$$\begin{aligned} &\left(\sum_{j=1}^{n} h_{jj}^{n+1}\right)^2 - 2\sum_{j=1}^{n}\left(h_{jj}^{n+1}\right)^2 \\ &= 2\eta + 4\sum_{1\leq i<j\leq n}\left(h_{ij}^{n+1}\right)^2 + 2\sum_{r=n+2}^{2m}\sum_{i,j=1}^{n}\left(h_{ij}^{r}\right)^2. \end{aligned} \tag{3.3}$$

Because (n_1, n_2) is a partition of $n_1 + n_2$, Lemma 1 implies that

$$\begin{aligned} &\sum_{1\leq\alpha<\beta\leq n_1} 4h_{\alpha\alpha}^{n+1}h_{\beta\beta}^{n+1} + \sum_{n_1+1\leq s<t\leq n} 4h_{ss}^{n+1}h_{tt}^{n+1} \\ &\geq \left(\sum_{j=1}^{n} h_{jj}^{n+1}\right)^2 - 2\sum_{j=1}^{n}\left(h_{jj}^{n+1}\right)^2, \end{aligned} \tag{3.4}$$

with the equality holding if and only if

$$\sum_{\alpha=1}^{n_1} h_{\alpha\alpha}^{n+1} = \sum_{s=n_1+1}^{n} h_{ss}^{n+1}. \tag{3.5}$$

Combining (3.3) and (3.4) gives

$$\begin{aligned} &\sum_{1\leq\alpha<\beta\leq n_1} h_{\alpha\alpha}^{n+1}h_{\beta\beta}^{n+1} + \sum_{n_1+1\leq s<t\leq n} h_{ss}^{n+1}h_{tt}^{n+1} \\ &\geq \frac{\eta}{2} + \sum_{1\leq j<k\leq n}\left(h_{jk}^{n+1}\right)^2 + \frac{1}{2}\sum_{r=n+2}^{2m}\sum_{j,k=1}^{n}\left(h_{jk}^{r}\right)^2, \end{aligned} \tag{3.6}$$

with the equality holding if and only if (3.5) occurs.

On the other hand, (2.6) and (2.11) imply

$$\begin{aligned} \frac{n_2\Delta f}{f} &= \tau - \frac{n_1(n_1-1)}{2}c - \frac{n_2(n_2-1)}{2}c \\ &- \sum_{r=n+1}^{2m}\sum_{\alpha<\beta}\left(h_{\alpha\alpha}^r h_{\beta\beta}^r - (h_{\alpha\beta}^r)^2\right) \\ &- \sum_{r=n+1}^{2m}\sum_{s<t}\left(h_{ss}^r h_{tt}^r - (h_{st}^r)^2\right) \\ &- \sum_{\alpha<\beta} 3c\left\langle Pe_\alpha, e_\beta\right\rangle^2 - \sum_{s<t} 3c\left\langle Pe_s, e_t\right\rangle^2. \end{aligned} \tag{3.7}$$

Therefore, by (3.1), (3.6) and (3.7), we find

$$\begin{aligned} \frac{n_2\Delta f}{f} &\leq \tau - \frac{n(n-1)}{2}c + n_1n_2c - \frac{\eta}{2} \\ &- \sum_{r=n+1}^{2m}\sum_{\alpha,t}\left(h_{\alpha t}^r\right)^2 - \frac{1}{2}\sum_{r=n+2}^{2m}\left(\sum_{\alpha} h_{\alpha\alpha}^r\right)^2 \\ &- \frac{1}{2}\sum_{r=n+2}^{2m}\left(\sum_{t} h_{tt}^r\right)^2 - \sum_{\alpha<\beta} 3c\left\langle Pe_\alpha, e_\beta\right\rangle^2 \\ &- \sum_{s<t} 3c\left\langle Pe_s, e_t\right\rangle^2. \end{aligned} \tag{3.8}$$

Hence, we have

$$\begin{aligned}\frac{n_2\Delta f}{f} &\le \tau - \frac{n(n-1)}{2}c + n_1n_2c - \frac{\eta}{2} \\ &- \sum_{\alpha<\beta} 3c\,\langle Pe_\alpha, e_\beta\rangle^2 - \sum_{s<t} 3c\,\langle Pe_s, e_t\rangle^2\end{aligned} \tag{3.9}$$

with the equality holding if and only if ϕ is mixed totally geodesic and

$$\sum_{\alpha=1}^{n_1} h^r_{\alpha\alpha} = \sum_{t=n_1+1}^{n} h^r_{tt} = 0 \tag{3.10}$$

for $r = n+1, \ldots, 2m$. Combining (3.1) and (3.9) yields

$$\begin{aligned}\frac{\Delta f}{f} &\le \frac{n^2}{4n_2}H^2 + n_1c + \frac{3c}{n_2}\sum_{\alpha,t}\langle Pe_\alpha, e_t\rangle^2 \\ &\le \frac{n^2}{4n_2}H^2 + n_1c + 3c\min\left\{\frac{n_1}{n_2}, 1\right\}.\end{aligned} \tag{3.11}$$

In particular, if $\phi : N_1 \times_f N_2 \to CP^m(4c)$ is totally real, (3.11) implies that

$$\frac{\Delta f}{f} \le \frac{n^2}{4n_2}H^2 + n_1c. \tag{3.12}$$

Now, we divide the proof into two cases.

Case (a): $n_1 \le n_2$. In this case, (3.11) implies that

$$\frac{\Delta f}{f} \le \frac{n^2}{4n_2}H^2 + (n_1+3)\,c. \tag{3.13}$$

Suppose that the equality case of (3.13) holds identically. Then we have

(a.1) $n_1 = n_2$,

(a.2) $J\mathcal{H} = \mathcal{V}$, and

(a.3) the immersion is mixed totally geodesic.

From (a.2) we know that $N_1 \times_f N_2$ is immersed as a complex submanifold. Hence, we obtain from conditions (a.2) and (a.3) that

$$h(X,Y) = -Jh(X,JY) = 0 \tag{3.14}$$

for $X, Y \in \mathcal{H}$.

Similarly, we also have

$$h(Z,W) = 0 \quad \text{for} \quad Z, W \in \mathcal{V}. \tag{3.15}$$

By combining (3.14) and (3.15) with (a.3), we know that the warped product is also totally geodesic. Hence, it is immersed as an open part of $CP^{n_1}(4c)$. Also, (a.2) implies that leaves and fibers of $N_1 \times_f N_2$ are immersed as Lagrangian submanifolds. By the fact that leaves are totally geodesic Lagrangian submanifolds of $CP^{n_1}(4c)$, we also know that N_1 is isometric to an open part of a real projective n_1-space $RP^{n_1}(1)$ of constant curvature one.

On the other hand, since fibers are totally umbilical in $N_1 \times_f N_2$, they are totally umbilical Lagrangian submanifolds in $CP^{n_1}(4c)$. Hence, by applying Theorem 1 of [5], we conclude that either

(i) $n_1 = n_2 = 1$, or

(ii) fibers are totally geodesic in $N_1 \times_f N_2$.

If case (ii) occurs, then f is constant. But this cannot happen, since $CP^{n_1}(4c)$ is locally irreducible. So, we must have $n_1 = n_2 = 1$.

Since $N_1 \times_f N_2$ is totally geodesic in $CP^m(4c)$ and $n_1 = 1$, the equality case of (1.2) implies that f is an eigenfunction of Δ with eigenvalue $4c$.

The converse is easy to verify.

Case (b): $n_1 > n_2$. In this case, (3.11) gives

$$\frac{\Delta f}{f} \le \frac{n^2}{4n_2}H^2 + (n_1+3)\,c \tag{3.16}$$

with equality holding if and only if we have

(b.1) $J\mathcal{V} \subset \mathcal{H}$,

(b.2) ϕ is mixed totally geodesic, and

(b.3) $\operatorname{tr} h_1 = \operatorname{tr} h_2 = 0$.

Now, assume that the equality sign of (3.16) holds identically.

For vertical vector fields Z, W in $\mathcal{V}$, we have $\tilde\nabla_{JZ}JW = J\tilde\nabla_{JZ}W$. Hence, (b.1), (b.2) and the formulas of Gauss and Weingarten imply that

$$\nabla_{JZ}(JW) + h(JZ, JW) = J\nabla_{JZ}W. \tag{3.17}$$

On the other hand, since leaves are totally geodesic in $N_1 \times_f N_2$, $\nabla_{JZ}W$ is always tangent to fibers for vertical vector fields Z, W. So, $J\nabla_{JZ}W$ is tangent to leaves according to (b.1). Thus, we obtain from (3.17) that

$$h(J\mathcal{V}, J\mathcal{V}) = \{0\}. \tag{3.18}$$

Assume that $n_2 > 1$. Let (p,q) be a fixed point in $N_1 \times N_2$ and let Z_1, Z_2 be two orthogonal nonzero vector fields in $\mathcal{D}_2$ with $|Z_1| = |Z_2|$. We choose a vector field X in $\mathcal{D}_1$ such that $X = JZ_1$ at (p,q) Since we have $h(\mathcal{H},\mathcal{V}) = h(J\mathcal{V}, J\mathcal{V}) = \{0\}$ and $\langle X, Z_2\rangle = \langle X, JZ_2\rangle = 0$ at (p,q), equation (2.6) of Gauss implies that

$$K(X, Z_1) = 4K(X, Z_2) \quad \text{at} \quad (p,q). \tag{3.19}$$

On the other hand, from (2.10) we have $K(X,Z_1) = K(X,Z_2)$ which contradicts to (3.19). Hence, we must have $n_2 = 1$.

Let

$$\mathcal{H} = \mathcal{L} \oplus J\mathcal{V}$$

be an orthogonal decomposition of $\mathcal{H}$. Since the rank of $J\mathcal{V}$ is one, there is a unit vector field η in $J\mathcal{V}$.

For any horizontal vector $X \in \mathcal{H}$, we obtain from (b.2) that

$$J\nabla_X \eta + Jh(X, \eta) = \nabla_X(J\eta). \tag{3.20}$$

Because $n_2 = 1$, the leaves are totally geodesic in $N_1 \times_f N_2$, and $J\eta$ is a unit vector normal field of the leaves, so Weingarten's formula gives

$$\nabla_X(J\eta) = -A^1_{J\eta}X + D^1_X J\eta = 0, \tag{3.21}$$

where A^1 and D^1 denote the shape operator and the normal connection of leaves in $N_1 \times_f N_2$.

Combining (3.20) and (3.21) gives

$$\nabla_X \eta = 0, \tag{3.22}$$

$$h(X, \eta) = 0 \tag{3.23}$$

for $X \in \mathcal{H} = \mathcal{L} \oplus J\mathcal{V}$.

Equation (3.22) implies that both $\mathcal{L}$ and $J\mathcal{V}$ are totally geodesic distributions. Hence, locally N_1 is the Riemannian product $L \times I$, where L and I are integral submanifolds of $\mathcal{L}$ and $J\mathcal{V}$, respectively.

Choose a unit speed geodesic $\gamma = \gamma(s)$ in L. Let us consider the immersion:

$$\hat{\phi} : \gamma \times I \times N_2 \xrightarrow{\text{inclusion}} N_1 \times_f N_2 \xrightarrow{\phi} CP^m(4c).$$

With respect to the induced metric, $\gamma \times I \times N_2$ is also a warped product manifold $\gamma \times I \times_{\hat{f}} N_2$, where $\hat{f}$ is the restriction of f on $\gamma \times I$.

Let σ denote the second fundamental form of $\gamma \times I \times_{\hat{f}} N_2$ in $N_1 \times_f N_2$ and let $\hat{h}, \hat{A}, \ldots$, etc., be the second fundamental form, the shape operator, ..., etc., of $\gamma \times I \times_{\hat{f}} N_2$ in $CP^m(4c)$, respectively. Then we have

$$\hat{h}(x, y) = h(x, y) + \sigma(x, y) \tag{3.24}$$

for x, y tangent to $\gamma \times I \times_{\hat{f}} N_2$. Since γ is a geodesic in L, Lemma 9 of [6] gives

$$\begin{aligned} \sigma(\gamma', \eta) &= \sigma(\gamma', J\eta) = \sigma(\eta, \eta) \\ &= \sigma(\eta, J\eta) = 0. \end{aligned} \tag{3.25}$$

From (b.2), (3.17) and (3.23)–(3.25) we get

$$\begin{aligned} \hat{h}(\gamma', \eta) &= \hat{h}(\gamma', J\eta) = \hat{h}(\eta, \eta) \\ &= \hat{h}(\eta, J\eta) = 0. \end{aligned} \tag{3.26}$$

Using (2.3) and (3.26) we find

$$\begin{aligned} &(\bar{\nabla}_\eta \hat{h})(J\eta, \gamma') - (\bar{\nabla}_{J\eta}\hat{h})(\eta, \gamma') \\ &\quad = \hat{h}(\eta, \hat{\nabla}_{J\eta}\gamma') - \hat{h}(J\eta, \hat{\nabla}_\eta \gamma'), \end{aligned} \tag{3.27}$$

where $\hat{\nabla}$ is the Levi-Civita connection of $\gamma \times I \times_{\hat{f}} N_2$.

Equations (2.4), (2.5), (3.26), and (3.27) imply that

$$\begin{aligned} 2c &= \tilde{R}(\eta, J\eta; \gamma', J\gamma') \\ &= \langle \hat{h}(\eta, \hat{\nabla}_{J\eta}\gamma'), J\gamma' \rangle - \langle \hat{h}(J\eta, \hat{\nabla}_\eta \gamma'), J\gamma' \rangle \\ &= -\langle \hat{A}_{J\gamma'} J\eta, \hat{\nabla}_\eta \gamma' \rangle. \end{aligned} \tag{3.28}$$

On the other hand, from (2.1), (2.2), (3.26) and (3.27), we find

$$J\hat{\nabla}_\eta \gamma' = \tilde{\nabla}_\eta J\gamma' = \hat{D}_\eta J\gamma'. \tag{3.29}$$

Since $\hat{A}_{J\gamma'} J\eta \in \operatorname{Span}\{J\eta\}$ by (3.26), (3.29) implies

$$\left\langle \hat{A}_{J\gamma'} J\eta, \hat{\nabla}_\eta \gamma' \right\rangle = 0 \tag{3.30}$$

which contradicts to (3.28) due to the fact: $c > 0$. Hence, case (b) cannot occur. This completes the proof of Theorem 1.

Theorem 2 is an immediate consequence of inequality (1.2).

For the proof of Theorem 3, let us assume that f is a positive function on a Riemannian n_1-manifold such that

$$\frac{\Delta f}{f} > n_1 \tag{3.31}$$

at some point $p \in N_1$ and let N_2 be an arbitrary Riemannian manifold of positive dimension. If $N_1 \times_f N_2$ admits an isometric totally real minimal immersion into $CP^m(4)$, then (3.12) implies that

$$\frac{\Delta f}{f} \le n_1 \tag{3.32}$$

at every point in N_1 which contradicts to (3.31). This proves Theorem 3. □

4. Examples.

Example 1. Let $I = (-\pi/4, \pi/4)$, $N_2 = S^1(1)$ and $f = (1/2)\cos 2s$. Then the warped product

$$N_1 \times_f N_2 =: I \times_{(\cos 2s)/2} S^1(1)$$

has constant sectional curvature 4. Clearly, we have $(\Delta f)/f = 4$. If we define the complex structure J on the warped product by

$$J\left(\frac{\partial}{\partial s}\right) = 2(\sec 2s)\frac{\partial}{\partial t}, \tag{4.1}$$

then $(I \times_{(\cos 2s)/2} S^1(1), g, J)$ is holomorphically isometric to a dense open subset of $CP^1(4)$.

Let $\phi : CP^1(4) \to CP^m(4)$ be a standard totally geodesic embedding of $CP^1(4)$ into $CP^m(4)$. Then the restriction of ϕ to $I \times_{(\cos 2s)/2} S^1(1)$ gives rise to a minimal isometric immersion of $I \times_{(\cos 2s)/2} S^1(1)$ into $CP^m(4c)$ which satisfies the equality case of (1.2) on $I \times_{(\cos 2s)/2} S^1(1)$ identically.

Example 2. Consider the same warped product $N_1 \times_f N_2 = I \times_{(\cos 2s)/2} S^1(1)$ as given in Example 1. Let $\phi : CP^1(4) \to CP^m(4)$ be the totally geodesic holomorphic embedding of $CP^1(4)$ into $CP^m(4)$. Then the restriction of ϕ to $N_1 \times_f N_2$ is an isometric minimal immersion of $N_1 \times_f N_2$ into $CP^m(4)$ which satisfies $(\Delta f)/f = 3 + n_1$ identically.

This example shows that the assumption "$(\Delta f)/f > 3 + n_1$ at some point in N_1" given in Theorem 2 is best possible.

Example 3. Let $S^{n-1}(1)$ denote the unit $(n-1)$-sphere and g_1 be the standard metric on $S^{n-1}(1)$. Denote by $N_1 \times_f N_2$ the warped product given by $N_1 = (-\pi/2, \pi/2)$, $N_2 = S^{n-1}(1)$ and $f = \cos s$. Then the warped function of this warped product satisfies

$$\frac{\Delta f}{f} = n_1 \tag{4.2}$$

identically. Moreover, it is easy to verify that this warped product is isometric to a dense open subset of $S^n(1)$.

Let

$$\phi : S^n(1) \xrightarrow[\text{2:1}]{\text{projection}} RP^n(1) \xrightarrow[\text{totally real}]{\text{totally geodesic}} CP^n(4)$$

be a standard totally geodesic Lagrangian immersion of $S^n(1)$ into $CP^n(4)$. Then the restriction of ϕ to $N_1 \times_f N_2$ is a totally real minimal immersion.

This example illustrates that the assumption "$(\Delta f)/f > n_1$ at some point in N_1" given in Theorem 3 is also sharp.

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