General Hodge conjecture for abelian varieties of CM-type

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Abstract: The general Hodge conjecture for abelian varieties of CM-type is shown to be implied by the usual Hodge conjecture for those up to codimension two.

Key words: Hodge conjecture; abelian variety.

1. Introduction. The purpose of this article is to announce that the validity of the Hodge conjecture in codimension two implies that of the whole general Hodge conjecture (GHC for short) for any abelian varieties of CM-type. The main ingredient is the theory of abelian varieties associated to hyperplane arrangements as is developed in [3]. In particular, the notion of "*N-dominatedness*" introduced in [3] plays an essential role for us to understand what kind of exceptional Hodge cycles should be proved to be algebraic. Our strategy for the proof goes roughly as follows: Given a Galois CM-field K with $\mathrm{Gal}(K/\mathbf{Q}) \cong G$, we associate an abelian variety $A_{\mathcal{A}(2^n)}(G;K)$ to a hyperplane arrangement $\mathcal{A}(2^n)$ in $\mathbf{R}^n$. Thereafter we show an arbitrary abelian variety A of CM-type split by K can be embedded into an appropriate self-product $A_{\mathcal{A}(2^n)}(G;K)^m$ (Proposition 6.2). Thus GHC for A is reduced to GHC for $A_{\mathcal{A}(2^n)}(G;K)^m$ (Lemma 2.1). Furthermore we reduce GHC for $A_{\mathcal{A}(2^n)}(G;K)^m$ to the usual Hodge conjecture for $A_{\mathcal{A}(2^n)}(G;K)$ by translating the properties of various rational sub-Hodge structures of its cohomology spaces into some combinatorial properties of the arrangement $\mathcal{A}(2^n)$. Thus the fact that $A_{\mathcal{A}(2^n)}(G;K)$ is 2-*dominated* (Theorem 7.4) implies the aforementioned result. Details will appear elsewhere.

2. General Hodge conjecture. Let X be a smooth projective variety over $\mathbf{C}$. For any rational sub-Hodge structure $W \subset H^k(X,\mathbf{Q})$, we define the level $l(W)$ by $l(W) = \max\{p-q; W_{\mathbf{C}}^{p,q} \neq 0\}$. Then GHC for X is formulated as follows:

For any rational sub-Hodge structure $W \subset H^k(X,\mathbf{Q})$ *with* $l(W) = k-2p$, *there exists a Zariski-closed subset* Z *of codimension* p *on* X *such that* $W \subset \ker\{H^k(X,\mathbf{Q}) \to H^k(X-Z,\mathbf{Q})\}$.

The following lemma is crucial in view of Proposition 6.3 below.

Lemma 2.1. *Let A be an abelian variety and B an abelian subvariety of A. Then GHC for A implies that for B.*

3. Varchenko and Hodge matrices. Let $\mathcal{A} = \{H_1,\dots,H_k\}$ be a hyperplane arrangement in $\mathbf{R}^n$, and let $R(\mathcal{A}) = \{R_1,\dots,R_m\}$ denote the set of regions of the complement of the union of $\mathcal{A}$. For regions $S,T \in R(\mathcal{A})$ the number of hyperplanes in $\mathcal{A}$ which separate S and T is denoted by $d(S,T)$. We introduced in [3] the matrix

$$D = D(\mathcal{A}) = (d(S,T))_{(S,T)\in R(\mathcal{A})\times R(\mathcal{A})},$$

and called it *the additive version of Varchenko matrix* (abbreviated as *AV-matrix*), the rows and columns being ordered according to the given numbering of $R(\mathcal{A})$. Let $V(\mathcal{A}) = \left\{\sum_{R\in R(\mathcal{A})} a_R R; a_R \in \mathbf{Q}\right\}$ be the $\mathbf{Q}$-vector space consisting of the formal $\mathbf{Q}$-linear combinations of the elements in $R(\mathcal{A})$.

Proposition 3.1 (see [3, Proposition 2.1]). *For any hyperplane $H \in \mathcal{A}$, let $h_H \in V(\mathcal{A})$ denote the vector whose j-th entry $(h_H)_j$ is defined by the rule*

$$(h_H)_j = \begin{cases} 1, & \text{if } H \text{ does not separate } R_j \text{ and } R_1, \\ -1, & \text{otherwise.} \end{cases}$$

Let Row-sp$(D(\mathcal{A}))$ *denote the subspace of* $V(\mathcal{A})$ *generated by the row vectors of* $D(\mathcal{A})$. *Then we have* $\text{Row-sp}(D(\mathcal{A})) = \langle h_H; H \in \mathcal{A}, \mathbf{1}\rangle_{\mathbf{Q}}$, *where* $\mathbf{1} = \sum_{R\in R(\mathcal{A})} R \in V(\mathcal{A})$, *and* $\dim \text{Row-sp}(D(\mathcal{A})) = \sharp(\mathcal{A}) + 1$.

Remark. In [3, Proposition 2.1], we made an assumption that (O): *there exists a pair* $(R,S) \in R(\mathcal{A}) \times R(\mathcal{A})$ *such that* $d(R,S) = k$ $(= \sharp\mathcal{A})$, for the

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validity of the proposition. After the paper was published, however, Prof. Vojta kindly informed to the author that this assumption holds for any hyperplane arrangement.

Let $H(\mathcal{A})$ denote the k by m matrix consisting of k row vectors $h_{H_1}, \ldots, h_{H_k}$, and let $\mathbf{Hsp}(\mathcal{A})$ denote its row space. We call $H(\mathcal{A})$ *Hodge matrix* associated to the hyperplane arrangement $\mathcal{A}$. It follows from Proposition 3.1 that $\operatorname{rank} H(\mathcal{A}) = \dim \mathbf{Hsp}(\mathcal{A}) = k$.

4. Generalities on abelian varieties of CM-type. Let K be a Galois CM-extension of $\mathbf{Q}$ with $\mathrm{Gal}(K/\mathbf{Q}) \cong G$. Let $\rho \in G$ denote the complex conjugation.

(4.A) The category of CM $\mathbf{Q}$-algebras (= products of CM-fields) split by K is anti-equivalent to the category of finite G-sets by the correspondences, $F \mapsto \mathrm{Hom}_{\mathbf{Q}\text{-algebra}}(F, K)$ for F a CM $\mathbf{Q}$-algebra.

(4.B) A finite G-set S, plus the data of $S_1 \subset S$ with S the disjoint sum of S_1 and ρS_1, give an abelian variety A of CM type (up to isogeny), on which the CM-algebra F corresponding to S acts by endomorphisms: the rational lattice is $F \cong \mathrm{Hom}_G(S, K)$ and A is $\mathbf{C}^{S_1}/\{\text{a lattice in } F\}$.

Remark. For any set X and integer $n \geq 1$, we denote the disjoint union of n copies of X by $X_{(n)}$. We identify $X_{(n)}$ with $X \times [1, n]$, where $[1, n] = \{1, 2, \ldots, n\}$, and denote by p the natural projection $X_{(n)} = X \times [1, n] \to X$. In this notation, if an abelian variety A corresponds to $S_1 \subset S$ as in (4.B), then the self-product A^n, $n \geq 1$, corresponds to the disjoint union $(S_1)_{(n)} \subset S_{(n)}$ endowed with natural G-set structure.

The Hodge ring (= the ring of Hodge cycles) of A is described as follows (see [2, 3] for detail):

(4.C) The first cohomology group $H^1(A, \mathbf{C})$ can be identified with $\mathbf{C}^S$, and S_1 defines a one-parameter subgroup T of $\mathrm{Gl}(\mathbf{C}^S)$. The Hodge group $Hg(A)$ is given by T and its conjugates. As a consequence, the complexification of the Hodge ring ($\subset \Lambda\mathbf{C}^S$) admits as basis the set of basis vectors of $\Lambda\mathbf{C}^S$ corresponding to subsets P of S with the property that $\sharp(P \cap S_1 g) = (\sharp P)/2$ for any $g \in G$.

An abelian variety A of CM-type is said to be *N-dominated* if the following condition holds: *for every $n \geq 1$ the Hodge ring $\mathcal{H}(A^n)_{\mathbf{C}}$ of A^n is spanned by the Hodge classes $[P]$, $P \subset S_{(n)}$ with $\sharp(P) \leq 2N$.* Furthermore A is said to be *h-degenerate* if *for every $n \geq 1$ the Hodge ring $\mathcal{H}(A^n)_{\mathbf{C}}$ of A^n is spanned by the Hodge classes $[P]$, $P \subset S_{(n)}$ with $d(P)_s \leq h$, $s \in S$, where $d(P)_s = \sharp(p^{-1}(s) \cap P)$.*

In view of the following proposition, this notion plays an important role when we try to prove the Hodge conjecture.

Proposition 4.1. *Let A be an abelian variety of CM-type. Suppose that A is h-degenerate and the Hodge conjecture holds for any A^k, $k \leq h$. Then it holds for all self-products A^n, $n \geq 1$.*

5. Abelian varieties associated to CM-arrangements. A hyperplane arrangement $\mathcal{A}$ in $\mathbf{R}^n$ is said to be central when any hyperplane of $\mathcal{A}$ contains the origin of $\mathbf{R}^n$. Let K be a Galois CM-field with $\mathrm{Gal}(K/\mathbf{Q}) = G$.

Definition 5.1. A central hyperplane arrangement $\mathcal{A}$ in $\mathbf{R}^n$ is said to be a CM-arrangement with respect to the pair (G, K), if there exists an embedding $i : G \to \mathrm{GL}_n(\mathbf{R})$ such that $i(G)$ acts transitively on $\mathcal{A}$ and $i(\rho)$ is the multiplication by -1.

Given a CM-arrangement $\mathcal{A}$, and for any hyperplane $H \in \mathcal{A}$, let $H^{>0}$ denote the connected component of $\mathbf{R}^n - H$ which contains the region R_1, and $H^{<0}$ the other component. Let $H^+ = \{R \in R(\mathcal{A});\ R \subset H^{>0}\}$, $H^- = \{R \in R(\mathcal{A});\ R \subset H^{<0}\}$ $(= H^+\rho)$. In (4.B), we take the set $R(\mathcal{A})$ as G-set S, and the subset $H_1^+ \subset R(\mathcal{A})$ as $S_1 \subset S$, and denote by $A_{\mathcal{A}}(G; K)$ the abelian variety corresponding to the pair $(S, S_1) = (R(\mathcal{A}), H_1^+)$. Then the fact that G acts transitively on $\mathcal{A}$ implies the following proposition, which is rather unexpected.

Proposition 5.2. *The structure of the Hodge ring of $A_{\mathcal{A}}(G; K)$ as well as $A_{\mathcal{A}}(G; K)^n$, $n \geq 1$, depends only on $\mathcal{A}$, and not on the pair (G, K).*

A hyperplane arrangement $\mathcal{A}(2^n)$ in $\mathbf{R}^n$, called *the hyperplane arrangement of $(2, \ldots, 2)$-type*, is defined by $\mathcal{A}(2^n) = \{H_1, \ldots, H_n\}$, where $H_i = \{(x_1, \ldots, x_N) \in \mathbf{R}^n;\ x_i = 0\}$, $1 \leq i \leq n$. Each region of $\mathcal{A}(2^n)$ is specified by the sign of the coordinates: $R(\mathcal{A}(2^n)) = \{R(\varepsilon_1, \ldots, \varepsilon_n);\ \varepsilon_i \in \{\pm 1\}, 1 \leq i \leq n\}$, where $R(\varepsilon_1, \ldots, \varepsilon_n) = \{(x_1, \ldots, x_n) \in \mathbf{R}^n;\ \mathrm{sgn}(x_i) = \varepsilon_i,\ 1 \leq i \leq n\}$. In particular, we have $\sharp(R(\mathcal{A}(2^n))) = 2^n$.

6. Abelian varieties of CM-type and the hyperplane arrangement of $(2, \ldots, 2)$-type. Let K be a Galois CM-field of degree $2n$ and let $G = \mathrm{Gal}(K/\mathbf{Q})$. It follows from [1] that there is an injective homomorphism $\Phi : G \to \{\pm 1\} \operatorname{wr} S_n$. For later use we recall briefly the construction of Φ. Let $\rho \in G$ denote the complex conjugation, and let $S_1 =$

$\{g_1, \ldots, g_n\} \subset G$ be a CM-type of K so that $G = \{g_1, \ldots, g_n, g_1\rho, \ldots, g_n\rho\}$. We assume that g_1 is the identity of G. We define a map $\Sigma : G \to \{\pm 1\}^n$ by the rule

$$(6.1) \quad \Sigma(g) = (\varepsilon_1, \ldots, \varepsilon_n) \in \{\pm 1\}^n,$$

$$\text{where } \varepsilon_i = \begin{cases} 1, & \text{if } g \in S_1 g_i^{-1}, \\ -1, & \text{if } g \notin S_1 g_i^{-1}, \end{cases}$$

for $1 \le i \le n$. Hence for any $g \in G$, there exists a unique permutation $\sigma = \Pi(g) \in S_n$ such that $gg_i = \rho^{\mu(\varepsilon_i)} g_{\sigma^{-1}(i)}$, where the map $\mu : \{\pm 1\} \to \mathbf{Z}/2\mathbf{Z}$ is the homomorphism defined by $\mu(1) = 0$, $\mu(-1) = 1$.

Proposition 6.1. *Notation being as above, let $\Phi : G \to \{\pm 1\} \operatorname{wr} S_n$ be the map defined by $\Phi(g) = (\Sigma(g); \Pi(g))$ for any $g \in G$. Then we have the following:*

(i) *Φ is an injective homomorphism,*

(ii) *the image of $\Phi(G)$ under the natural projection $\{\pm 1\} \operatorname{wr} S_n \to S_n$ is a transitive subgroup of S_n,*

(iii) *if A is simple then the map $\Sigma : G \to \{\pm 1\}^n$ is injective.*

We can consider the wreath product $G_n = \{\pm 1\} \operatorname{wr} S_n$ naturally as a subgroup of $\mathrm{GL}(\mathbf{R}^n)$ by the rule

$$(6.2) \quad (x_1, \ldots, x_n)((\varepsilon_1, \ldots, \varepsilon_n); \sigma) = (\varepsilon_1 x_{\sigma^{-1}(1)}, \ldots, \varepsilon_n x_{\sigma^{-1}(n)}).$$

Then the homomorphism Φ provides $\mathbf{R}^n$ with the structure of a right G-module. Note that the set of hyperplanes $\mathcal{A}(2^n)$ is stable under this action of G_n on $\mathbf{R}^n$, and that, by Proposition 6.1 (ii), G acts transitively on $\mathcal{A}(2^n)$. Hence $\mathcal{A}(2^n)$ is a CM-arrangement with respect to (G, K) in the sense of Section four. By using this, we can show the following:

Proposition 6.2. *Any simple abelian variety of CM-type with Galois CM-field K such that $\mathrm{Gal}(K/\mathbf{Q}) \cong G$ is realized as a simple component of $A_{\mathcal{A}(2^n)}(G; K)$.*

Since every abelian variety split by K is isogenous to the product of a number of simple components of abelian varieties of type (K, S_1) with suitable S_1's, we have the following:

Proposition 6.3. *Any abelian variety split by K is realized up to isogeny as an abelian subvariety of a certain self-product $A_{\mathcal{A}(2^n)}(G; K)^m$ for suitable $m \ge 1$.*

7. Kernel of the Hodge matrix for $\boldsymbol{\mathcal{A}(2^n)}$. Since the structure of the Hodge ring of $A_{\mathcal{A}(2^n)}(G; K)^m$, $m \ge 1$, does not depend on the pair (G, K) by Proposition 5.2, we can take any (G, K) for the investigation of the Hodge ring under the assumption that $\mathcal{A}(2^n)$ is a CM-arrangement with respect to (G, K). Accordingly we set $G = \{\pm 1\} \operatorname{wr} S_n$. Note that it has a central subgroup $\{((\varepsilon_1, \ldots, \varepsilon_n); e); \varepsilon_i \in \{\pm 1\}, i \in [1, n]\}$, which is isomorphic to $B_n = (\mathbf{Z}/2\mathbf{Z})^n$. Let K be a Galois CM-field with $\mathrm{Gal}(K/\mathbf{Q}) \cong G$ such that the complex conjugation corresponds to $\mathbf{1} = (1, \ldots, 1) \in B_n$. We let G act on $\mathbf{R}^n$ as in (6.2). Then $\mathcal{A}(2^n)$ is a CM-arrangement with respect to the pair (G, K). Since the action of B_n on the set $R(\mathcal{A}(2^n))$ of regions is simple and transitive, we can identify $R(\mathcal{A}(2^n))$ with B_n by the rule $(a_i) \in B_n$ *corresponds to* $R((-1)^{a_1}, \ldots, (-1)^{a_n})$. Hence we can identify $R(\mathcal{A}(2^n))$ with B_n. Under this identification the function $d(\cdot, \cdot)$ introduced in Section three coincides with the so-called *Hamming distance*. Furthermore the $\mathbf{Q}$-vector space $V(\mathcal{A}(2^n))$ spanned by $R(\mathcal{A}(2^n))$ is isomorphic as representation space of B_n to the group algebra $\mathbf{Q}[B_n]$.

Definition 7.1. For any $\mathbf{a} = (a_i) \in B_n$, let $\chi_{\mathbf{a}} \in \mathrm{Hom}(B_n, \mathbf{C}^*)$ denote the character of B_n defined by $\chi_{\mathbf{a}}(\sigma) = (-1)^{\sum_{1 \le i \le n} a_i \sigma_i}$ for $\sigma = (\sigma_i) \in B_n$. Let $v_{\mathbf{a}}$ denote the vector in $V(\mathcal{A}(2^n))$ $(= \mathbf{Q}[B_n])$ defined by $v_{\mathbf{a}} = \sum_{\sigma \in B_n} \chi_{\mathbf{a}}(\sigma)\sigma$.

The element $v_{\mathbf{a}}$ gives a basis element of the one-dimensional vector space $V(\chi_{\mathbf{a}}) = \{v \in V(\mathcal{A}(2^n)); \sigma v = \chi_{\mathbf{a}}(\sigma) v$ *for any* $\sigma \in B_n\}$, which affords the representation $\chi_{\mathbf{a}}$ of B_n, so that we have $V(\mathcal{A}(2^n)) = \bigoplus_{\mathbf{a} \in B_n} V(\chi_{\mathbf{a}})$.

Proposition 7.2. *As representation spaces of B_n, we have*

$$\text{Row-sp}(D(\mathcal{A}(2^n))) = \bigoplus_{1 \le i \le n} V(\chi_{\mathbf{e}_i}) \oplus V(\chi_{(0,\ldots,0)}),$$

$$\mathbf{Hsp}(\mathcal{A}(2^n)) = \bigoplus_{1 \le i \le n} V(\chi_{\mathbf{e}_i}).$$

Let $\rho : B_n \to B_n$ denote the map defined by $\rho(\mathbf{a}) = \mathbf{1}.\mathbf{a}$.

Theorem 7.3. *Let $B_n^0 = \{\mathbf{a} \in B_n; a_1 = 0\}$. For any $\mathbf{a} \in B_n^0$, let $d_{\mathbf{a}} = \mathbf{a} + \rho(\mathbf{a}) \in V(\mathcal{A}(2^n))$. For any pair (i, j) with $1 \le i < j \le n$, let $z_{ij} = \mathbf{0} + \rho(\mathbf{e}_i) + \rho(\mathbf{e}_j) + \mathbf{e}_{ij} \in V(\mathcal{A}(2^n))$, where*

$$\mathbf{0} = (0, \ldots, 0),$$

$$\mathbf{e}_{ij} = (0, \ldots, 0, \overset{i}{\check{1}}, 0, \ldots, 0, \overset{j}{\check{1}}, 0, \ldots, 0) \in B_n.$$

Then the kernel of the Hodge matrix $H(\mathcal{A}(2^n))$ is spanned by $\sigma d_{\mathbf{a}}$, with $\mathbf{a} \in B_n^0$, $\sigma \in B_n$, and σz_{ij}, with $1 \le i < j \le n$, $\sigma \in B_n$.

Thus by Proposition 4.9 in [3] we obtain the following:

Theorem 7.4. *When* $n = 1, 2$*, the abelian variety* $A_{\mathcal{A}(2^n)}(G; K)$ *is* 1*-dominated, namely, non-degenerate. When* $n \geq 3$*, the abelian variety* $A_{\mathcal{A}(2^n)}(G; K)$ *is* 1*-degenerate and* 2*-dominated.*

Corollary 7.4.1. *If every Hodge cycle of codimension two on* $A_{\mathcal{A}(2^n)}(G; K)$ *is algebraic, then the whole Hodge conjecture holds true for any self-products* $A_{\mathcal{A}(2^n)}(G; K)^m$, $m \geq 1$.

Thus an argument similar to that for [3] implies the following:

Theorem 7.5. *Suppose that the Hodge cycles of codimension two on* $A_{\mathcal{A}(2^n)}(G; K)$ *is algebraic. Then the whole GHC holds for any self-product* $A_{\mathcal{A}(2^n)}(G; K)^m$.

Combining this with Proposition 6.3 and Lemma 2.1, we obtain the following:

Theorem 7.6. *Suppose that every Hodge cycle of codimension two on* $A_{\mathcal{A}(2^n)}(G; K)$ *is algebraic for any pair* (G, K)*. Then the whole GHC holds for any abelian variety of CM-type.*

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