Invariants associated with blow-analytic homeomorphisms

By Laurentiu PAUNESCU

School of Mathematics and Statistics, University of Sydney, NSW 2006, Australia

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Abstract: Blow-analytic equisingularity is a relatively young branch of mathematics, it has been developed over the last three decades. This paper answers some fundamental questions raised by Kuo-Milman [10] and Koike (private communication). We believe that the results we present here are of interest for those working in blow-analytic equisingularity. In particular we show that the singular loci of analytic functions germs, correspond even under arc-analytic equivalences. We also give easier proofs of some results in [1].

Key words: Blow-analytic; arc-analytic.

1. Blow-analytic category. The main difficulty concerning the *blow-analytic category*, as far as calculus is concern, comes from the facts that it is closed neither under differentiation nor under integration, and also there is no global chain rule. For instance the following blow-analytic homeomorphism, see [16], has its jacobian matrix with all entries non arc-analytic functions (so all the partial derivatives of its components are no longer in the category).

Example 1.1. Let $h : (\mathbf{R}^3, C) \to (\mathbf{R}^3, h(C))$ be the map-germ defined by

$$(x, y, z) \mapsto \big(x + f(x + y, z),\ y + x + f(x + y, z),\ z + x + f(x + y, z)\big)$$

where

$$f(u, v) = \frac{uv^5}{u^4 + v^6} \ \text{ and } \ C = \{(x, -x, 0), x \in \mathbf{R}\}.$$

Their study involves essentially the resolution of singularities (see for instance [6] and also [2] for a recent proof).

For the reader's convenience, we recall here some basic notions related to the blow-analytic category (see [3–5, 8, 9, 12, 14]).

Let U be a neighbourhood of the origin of $\mathbf{R}^n$, M a real analytic manifold and $\pi : M \to U$ be a proper analytic real modification whose complexification (see [7]) is also a proper modification (we often simply say that "π is a modification"). For instance $x \to x^3$ is not a modification in our sense.

We say that $f : U \to \mathbf{R}^m$ is *blow-analytic* via π if $f \circ \pi$ has an analytic extension on M.

We say that f is *blow-analytic* if it does so via some modification. It will follow that *blow-analytic* maps are analytic outside a thin set. They may not be even differentiable maps.

We say that f is *blow-meromorphic via* π if $f \circ \pi$ can be written as a meromorphic map on M.

We say that f is *blow-meromorphic* if it does so via some modification. It will follow that *blow-meromorphic* maps are analytic except a thin set.

Let P be a function defined almost everywhere on U.

We say that P is a *blow-analytic unit* via a modification $\pi : M \to U$, if $P \circ \pi$ extends to an analytic function on M, which is a unit as an analytic function. It will follow that P and $1/P$ are bounded away from zero and also P has constant sign.

Let U_1, U_2 be two neighbourhoods of the origin of $\mathbf{R}^n$. We say that $h : U_1 \to U_2$ is a *blow-analytic homeomorphism* if $h : U_1 \to U_2$ is a homeomorphism and there is an analytic isomorphism $H : M_1 \to M_2$ so that $h \circ \pi_1 = \pi_2 \circ H$ for some modifications $\pi_i : M_i \to U_i$, $i = 1, 2$. In fact one can easily see that a homeomorphism $h : U_1 \to U_2$ is *blow-analytic homeomorphism* iff there is a modification-germ $\pi : M \to U_1$ such that $h \circ \pi$ is also a modification. In that case note that because h is homeomorphism it follows that π and $h \circ \pi$ have (set-wise) the same critical locus (here we assume that a modification has more to one points only in the critical locus).

We say that two analytic functions $f : U_1 \to \mathbf{R}$ and $g : U_2 \to \mathbf{R}$ are *blow-analytic equivalent* (*arc-analytic*) *if* $f = g \circ h$ *with* h *blow-analytic homeomorphism* (h, h^{-1} *arc-analytic homeomorphisms re-*

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spectively).

This is the minimum one should ask for a good notion of blow-analytic equisingularity. This paper points out some bad and also some good behaviour of blow-analytic equivalences defined as above. As a consequence of these facts we suggest a more restrictive definition, which looks closer to the definition of blow-analytic isomorphism given in [5]. It is not quite clear the relation between this last definition and the newly suggested one!

Let U be an open neighbourhood of the origin in $\mathbf{R}^n$ and let $f : U \to \mathbf{R}$ be an arc-analytic function, i.e., analytic along any analytic arcs (see [13]). It is easy to see that at each point of U we have well defined partial derivatives. However, in general, they are no longer arc-analytic functions. If moreover f is a blow-analytic function, then it is clear that its partial derivatives are analytic except on a thin set.

In order to understand the difficulties one encounters while dealing with blow-analytic category, it is important to mention the following fact.

Example 1.2. Indeed let $h : (\mathbf{R}^3, C) \to (\mathbf{R}^3, h(C))$ be the map-germ defined by (an altered version of 1.1)

$$(x, y, z) \mapsto \bigl(x + z^3, y + z^2, z - 2f(x + z^3, y + z^2)\bigr)$$

where

$$f(u, v) = \frac{uv^5}{u^4 + v^6} \quad \text{and } C = \{(-z^3, -z^2, z),\ z \in \mathbf{R}\}.$$

This is a blow-analytic homeomorphism and its jacobian is equal to 1 (in particular it is a blow-analytic unit). However surprisingly enough, along the curve $\{(0, 0, z), z \in \mathbf{R}\}$, which is even transversal to C, we have $(dh(0, 0, z)/dz) = (3z^2, 2z, 0)$!

2. Questions and answers.

2.1. Blow-analytic equivalences preserve singular loci. (i) (Satoshi Koike, private letter) *Recall that in the* [9] *definition of blow-analytic equivalence of f, $g : (R^n, 0) \to (R, 0)$, the centres NEED NOT be contained in the singular loci of $f^{-1}(0)$, $g^{-1}(0)$.*

Suppose f and g are blow-analytically equivalent in this sense. Is it true that if f has an isolated singularity at 0, then so does g? (Of course, $n > 2$.)

Regarding this question, we have the following fact, namely the singular locus should correspond via a blow-analytic equivalence, regardless the nature of the blow-analytic homeomorphism! This provides us with a new invariant of analytic functions with respect to blow-analytic equivalences.

The result follows from the following more general proposition.

Proposition 2.1. *Let $f : (R^n, 0) \to (R, 0)$ and $g : (R^m, 0) \to (R, 0)$ be two C^1 functions and $h : (R^n, 0) \to (R^m, 0)$ a map such that $g \circ h = f$ and $h(a + te_i)$ is differentiable in t, for any a near 0 and all $i = 1, 2, \ldots, n$. If Σ_f and Σ_g are the critical sets of f and g respectively, we have that $h^{-1}(\Sigma_g) \subset \Sigma_f$.*

Proof. By assumption $h(a + te_i)$ are differentiable curves $i = 1, 2, \ldots, n$. Therefore

$$\frac{d(g \circ h(a + te_i))}{dt}(0) = \sum_{j=1}^{n} \frac{\partial g}{\partial x_j}(h(a)) \frac{d(h_j(a + te_i))}{dt}(0)$$

and this is just

$$\frac{\partial f}{\partial x_i}(a).$$

This shows clearly that if $\operatorname{grad} g(h(a)) = 0$ then $\operatorname{grad} f(a) = 0$ as well, which proves our proposition. □

We note that a blow-analytic homeomorphism is arc-analytic, so therefore it satisfies the conditions of our proposition. This shows that the singular locus is preserved under arc-analytic equivalences.

Corollary 2.2. *Singular loci are preserved under arc-analytic equivalence.*

• However the analytic structures may be different as the following example shows (suggested by Toshizumi Fukui).

Example 2.3. $f_t(x, y) = x^4 + 2tx^2y^2 + y^4, t$ near 1.

This may suggest the following condition may be added to the old definition of blow-analytic equivalence of two analytic germs. If $f = g \circ h$, h blow-analytic homeomorphism, we may ask a kind of analytic rigidity, namely that the induced map $\Sigma_f \to \Sigma_g$ to be analytic isomorphism as well. Note that in the example above, the integration of the associated Kuo vector field gives a lipschitz trivialisation of the family. So it seems that the right notion of blow-analytic equivalence of two analytic germs must involve both analytic and geometric rigidity (geometric rigidity means only contact order preserving (which is less than lipschitz), for more details see [11]. From the discussion above it follows that these two conditions are independent.

We also note that the stronger condition, namely that the induced $(f^{-1}(0), \Sigma_f) \to (g^{-1}(0), \Sigma_g)$ is analytic isomorphism, creates moduli (see for example Whitney's family).

• On the other hand, a good thing about blow-analytic equivalences is the following. Assume that $f = g \circ h$ where $h \circ \pi_1 = \pi_2$ for some modifications $\pi_i : M_i \to U_i$, $i = 1, 2$. Let Σ_{π_i} denote the critical loci of π_i, $i = 1, 2$ (set-wise they are identical). If $\pi_1(\Sigma_{\pi_1}) \subset \Sigma_f$ then necessarily $\pi_2(\Sigma_{\pi_2}) \subset \Sigma_g$. Because h has to be analytic outside $\pi_1(\Sigma_{\pi_1})$, it follows that h is an analytic isomorphism outside Σ_f, so this condition should be required in a good definition of blow-analytic equivalence. In particular the above happens if π_1 makes f normal crossing. Note that if $\pi_1(\Sigma_{\pi_1}) \subset \Sigma_f$, then precisely $\Sigma_{f\circ\pi_1} = \pi_1^{-1}(\Sigma_f)$. In this case π_1 induces a morphism

$$((f \circ \pi_1)^{-1}(0), \Sigma_{f\circ\pi_1}) \to (f^{-1}(0), \Sigma_f).$$

If h induces an isomorphism $\Sigma_f \to \Sigma_g$ then it follows that Σ_{π_i}, $i = 1, 2$, coincide even as analytic spaces, which in turn will imply that the jacobian of h is a blow-analytic unit (compare [5]).

2.2. Multiplicities of analytic arcs and blow-analytic homeomorphisms. (i) (Satoshi Koike, private letter) In [16] is constructed examples of a blow-analytic homeomorphisms which do not preserve the multiplicity of arcs. However, those homeomorphisms trivialise a family of analytic functions with non-isolated singularities.

Can one construct an example of a t-parametrised family of ISOLATED singularities

$$f(x, y, z; t) : (R^3, 0) \to (R, 0)$$

which is blow-analytically trivial, but the trivialisation does not preserve the multiplicity of arcs.

The answer is affirmative.

Let us consider the following example.

$$\begin{aligned} h(x, y, z, u) &= \Big(x, y, z - \frac{x^3y^2}{x^4 + y^6}, u - \frac{y^7}{x^4 + y^6}\Big) \\ &= (a, b, c, d). \end{aligned}$$

This is a blow-analytic homeomorphism, and

$$h\Big(t^3, t^2, \frac{t}{2}, 0\Big) = (t^3, t^2, 0, t^2 \ldots).$$

However $ac + bd = f$ and $g = xz - y^2 + yu$ satisfy $f \circ h = g$.

Let

$$h_s(x, y, z, u) = \Big(x, y, z - s\frac{x^3y^2}{x^4 + y^6}, u - s\frac{y^7}{x^4 + y^6}\Big).$$

Then each h_s is a blow-analytic homeomorphism such that h_s is an analytic isomorphism outside (z, u)-plane. Let $f_s = xz + yu - sy^2$. Then each f_s has an isolated singularity at the origin and $f_s \circ h_s = f_0$. And also, f_s is linearly trivial.

(ii) *Can you find such a blow-analytic homeomorphism (as above) h, with the following property: h is an analytic isomorphism outside the origin.*

The answer is affirmative.

Let us consider the following examples.

$$h_s(x, y, z) = \Big(x, y, z - s\frac{x^3y^2}{x^4 + y^6 + z^{12}}\Big).$$

These are blow-analytic homeomorphisms (to show this, one may use the implicit function theorem [17]) which are analytic isomorphisms outside the origin, and do not preserve the order of arcs ($s \neq 0$).

Even more interesting is the following family.

$$\begin{aligned} h_s(x, y, z, u) = \Big(&x, y, z - s\frac{x^3y^2}{x^4 + y^6 + z^{12} + u^{12}}, \\ &u - s\frac{y(y^6 + z^{12} + u^{12})}{x^4 + y^6 + z^{12} + u^{12}}\Big) \\ = (&a, b, c, d). \end{aligned}$$

These are blow-analytic homeomorphisms (to show this, one may use twice the implicit function theorem [17]), and moreover

$$h_s(t^3, t^2, 0, 0) = \Big(t^3, t^2, \frac{-st}{2}, \frac{-st^4}{2}\Big).$$

However $f_s = xz - sy^2 + yu$ satisfy $f_s \circ h_s = f_0$. And also, f_s is linearly trivial. Note that the above examples satisfy the analytic rigidity but not the geometric one.

2.3. Special trivialisations. (i) (Kuo-Milman in [10]) *Is the weighted order preserved in the case of the trivialisations constructed in* [10]?

Our answer will be a bit more general, covering also the case Newton non-degenerate families, giving also an alternative proof of some of the results in [1] (for the weighted version see [15]).

Let us consider a non-degenerate Newton polygon $\mathcal{N}$ in $\mathbf{R}^n$, and $\mathcal{V}$ the set of vertices of $\mathcal{N}$. We define the following non-negative function associated to it, namely,

$$V(x) := \sum_{\alpha\in\mathcal{V}} x^{2\alpha}.$$

Definition 2.4. x^α lies above $\mathcal{N}$ if $\langle w_A, \alpha\rangle \geq d_A$ for all $n-1$ compact faces A of $\mathcal{N}$, where w_A is a weighted system determined by A, and d_A the corresponding degree of the quasihomogeneous polynomial determined by A. Similarly a polynomial lies above $\mathcal{N}$ if all its monomials lie above $\mathcal{N}$.

Note that since $\mathcal{N}$ is non-degenerate we have that all $w_{A,i} > 0$, $i = 1, \dots, n$.

Proposition 2.5. *x^α lies above $\mathcal{N}$ iff $|x^\alpha| \leq c\sqrt{V(x)}$, (written as $|x^\alpha| \lesssim \sqrt{V(x)}$), for some constant c, in a small neighbourhood of the origin.*

For each $i = 1, \dots, n$, we define the following characteristic exponents, $h_i := \sup_{A\in\mathcal{A}}(w_{A,i}/2d_A)$, where $\mathcal{A}$ represents the set of all $n-1$ faces of $\mathcal{N}$, and w_A is a weighted system determined by A, and d_A the corresponding degree.

Proposition 2.6. *Let P be a non-negative quasihomogeneous polynomial with respect to a system of weights w. The following are equivalent.*
(i) $P(x) > 0$, $x \neq 0$, $P(0) = 0$, *iff P is equivalent to ρ^{d_P} where $\rho(x) = (\sum x_i^{2p/w_i})^{1/2p}$ (see [15]), the corresponding 1-quasihomogeneous form with respect to the given system of weights $w = (w_1, \dots, w_n)$. Here d_P is the weighted degree of P.*
(ii) $|x^\alpha| \lesssim P(x)$ *iff* $|x^\alpha| \lesssim \rho^{d_P}(x)$ *iff* $\langle w, \alpha\rangle \geq d_P$.
We also have the following.
(iii) *A polynomial Q lies above $\mathcal{N}$ iff $|Q| \lesssim \sqrt{V(x)}$.*
(iv) $|Q| \lesssim V^d$ *iff* $V^{h_i}(\partial Q/\partial x_i) \lesssim V^d$, $i = 1, \dots, n$, *d a positive integer. In particular $V^{h_i}(\partial V/\partial x_i) \lesssim V$, $i = 1, \dots, n$.*

Let us assume that we have an analytic deformation $F(x,t) = f(x) + tG(x,t)$, of an analytic function f, such that $|\frac{\partial F}{\partial t}| \lesssim V^d$ and the gradient with respect to V,

$$\operatorname{grad}_V(F) := \sum_{i=1,n} V^{h_i}\frac{\partial F}{\partial x_i}\frac{\partial}{\partial x_i} + \frac{\partial F}{\partial t}\frac{\partial}{\partial t},$$

satisfies $||\operatorname{grad}_V(F)|| \gtrsim V^d$, d a positive integer, where $||\cdot||$ represents the standard norm.

Here we use the following notation,

$$\operatorname{grad}_{V,x}(F) := \sum_{i=1,n} V^{h_i}\frac{\partial F}{\partial x_i}\frac{\partial}{\partial x_i}.$$

In particular, under the assumptions above we also have that, $||\operatorname{grad}_{V,x}(F)|| \gtrsim V^d$.

We construct the following vector field.

$$\Phi(x,t) := \sum_{i=1,n} \Phi_i(x,t)\frac{\partial}{\partial x_i} - \frac{\partial}{\partial t},$$

where

$$\Phi_i(x,t) := \frac{(\partial F/\partial t)V^{2h_i}(\partial F/\partial x_i)}{||\operatorname{grad}_{V,x}(F)||^2} \lesssim V^{h_i}$$

(satisfied because of our assumptions).

If $\phi(x,t) = \phi_t(x)$ is obtained by integrating $\Phi(x,t)$ (it trivialises the family, see also [1] for another similar trivialisation), we have that $(d(V(\phi_t))/dt) \lesssim V(\phi_t))$, which in turn will imply (via an elementary differential equations argument) that $V(\phi(\gamma(s),t)) \lesssim V(\gamma(s))$ i.e., one has the following result, answering the question asked by Kuo-Milman in [10].

Proposition 2.7. *The homeomorphisms ϕ_t constructed as above, preserve the Newton order, i.e.,* $\operatorname{ord}_s(V(\phi_t(\gamma(s)) = \operatorname{ord}_s(V(\gamma(s))$.

We mention that under the assumptions above $\phi_t : (\mathbf{R}^n, \sqrt{V}) \to (\mathbf{R}^n, ||\cdot||)$ are lipschitz with respect to the specified norms. In particular if V induces the usual norm (homogeneous case) we recover the fact that when the initial homogeneous part is nondegenerate, the corresponding ϕ_t are bilipschitz. In the weighted case (or Newton polygon case) we cannot expect ϕ_t to be lipschitz with respect to the usual norms, not even with respect to the weighted norms.

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