## Isolation of the Weyl conformal tensor for Einstein manifolds

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**Abstract:** An isolation theorem of Weyl conformal tensor of positive Einstein manifolds is given, when its  $L^{n/2}$ -norm is small.

**Key words:** Einstein manifold; Weyl conformal tensor; Yamabe metric; Sobolev inequality.

1. Introduction. The aim of this note is to show that the Weyl conformal tensor of an oriented positive Ricci Einstein n-manifold,  $n \geq 4$  obeys the following isolation theorem.

**Theorem.** Let (M,g) be a compact, connected oriented Einstein n-manifold,  $n \geq 4$ , with positive scalar curvature s and of  $\operatorname{Vol}(g) = 1$ . Then, there exists a constant C(n), depending only on n such that if  $L^{n/2}$ -norm  $\|W\|_{L^{n/2}} < C(n) s$ , then W = 0 so that (M,g) is a finite isometric quotient of the standard n-sphere of unit volume.

This theorem generalizes the isolation theorem given by M. Singer [8].

**Theorem** [8]. Let (M,g) be a compact, connected oriented Einstein n-manifold  $(n = 2m \ge 4)$  with non-vanishing Euler characteristic  $\chi(M)$  and of positive scalar curvature. Then there is a constant  $\varepsilon > 0$ , depending on n and  $\chi(M)$ , such that if the  $L^{n/2}$ -norm  $\|W\|_{L^{n/2}} < \varepsilon$ , then W = 0 and so (M,g) is isometric to a quotient of  $S^n$  with the canonical metric up to constant.

Our theorem is valid even in odd dimension. For related results see [2].

The idea of the proof of our theorem is based on the Weitzenböck formula on the operator  $d_L: C^{\infty}(\Omega^1 \otimes \Omega^2) \longrightarrow C^{\infty}(\Omega^2 \otimes \Omega^2)$  exploited by the first author in [5] and also on a crucial use of the Sobolev inequality relating to Yamabe metrics.

**2. Sobolev inequality.** The Sobolev inequality of a compact Riemannian n-manifold  $(M,g), n \geq 3$ , can be described in terms of Yamabe metrics. Refer also to [1, 4, 7].

Consider the Yamabe functional  $Q_g: C^{\infty}_+(M) \longrightarrow \mathbf{R}$ ,

(1) 
$$f \mapsto Q_g(f) = \left\{ 4 \frac{n-1}{n-2} \int_M |\nabla f|^2 dv_g + \int_M s f^2 dv_g \right\} / \left\{ \int_M f^p dv_g \right\}^{(2/p)},$$

p = (2n/n - 2). The infimum of  $Q_g(f)$  is called the Yamabe constant  $\mu([g])$ , a conformal invariant of the conformal class [g];

(2) 
$$\mu([g]) = \inf\{Q_g(f) \mid f \in C_+^{\infty}(M)\}.$$

By the completely solved Yamabe problem there exists for any metric g an  $f \in C^{\infty}_{+}(M)$  satisfying  $Q_g(f) = \mu([g])$ . So, the metric  $\tilde{g} = f^{(4/n-2)}g$ , called the Yamabe metric, a conformal change of g has constant scalar curvature given by  $\tilde{s} = \mu([g]) \operatorname{Vol}(\tilde{g})^{-(2/n)}$ . Namely

(3) 
$$\mu([g]) = \tilde{s} \operatorname{Vol}(\tilde{g})^{(2/n)}.$$

We take a Yamabe metric in the conformal class [g], denoted by g and then obtain the Sobolev inequality

(4) 
$$4\frac{n-1}{n-2} \|\nabla f\|_{L^2}^2$$
  
 $\geq s \operatorname{Vol}(g)^{(2/n)} \{ \|f\|_{L^p}^2 - \operatorname{Vol}(g)^{-(2/n)} \|f\|_{L^2}^2 \},$   
 $f \in H_1^2(M)$ 

where p = (2n/n - 2).

Normalizing volume, we have

**Proposition 0.1.** Let g be a Yamabe metric on M in a conformal class and with unit volume. Then

(5) 
$$4\frac{n-1}{n-2}\|\nabla f\|_{L^{2}}^{2} \ge s\{\|f\|_{L^{p}}^{2} - \|f\|_{L^{2}}^{2}\},$$
$$f \in H_{1}^{2}(M).$$

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**Remark 0.2.** The inequality (5) holds when f is replaced by any tensor T because of the Kato's inequality

$$(6) |\nabla |T|| \le |\nabla T|.$$

Remark 0.3. Theorem 1.3 in [4] (see also Proposition 3.1, [7]) indicates that any Einstein metric must be Yamabe, provided it is not conformally flat

**3. Proof of Theorem.** We assume that W does not vanish identically. So the Einstein metric g is, from the above remark, a Yamabe metric in the conformal class [g].

We normalize g by constant rescaling so that Vol(g) = 1.

As observed in [8] by M. Singer the Weyl conformal tensor W satisfies

(7)

$$0 = (d_L \delta_L + \delta_L d_L)W = \nabla^* \nabla W + \frac{2s}{n}W + \{W, W\}.$$

Here  $\{\cdot,\cdot\}$  denotes a certain quadratic combination of W.

So we get

(8) 
$$\|\nabla W\|_{L^{2}}^{2} + \frac{2s}{n} \|W\|_{L^{2}}^{2} \le C_{n}^{-1} \|W\|_{L^{3}}^{3}$$
  
 $\le C_{n}^{-1} \|W\|_{L^{n/2}} \|W\|_{L^{p/2}}^{2}$ 

Here we used the Hölder inequality together with the pointwise inequality

$$(9) \qquad |\langle \{W, W\}, W\rangle| \le C_n^{-1} |W|^3$$

for a constant  $C_n > 0$ , depending only on n. Applying the Sobolev inequality in Proposition 0.1 yields

$$(10) C_n^{-1} \|W\|_{L^{n/2}} \|W\|_{L^p}^2$$

$$\geq \frac{2s}{n} \|W\|_{L^2}^2 + \frac{n-2}{4(n-1)} s(\|W\|_{L^p}^2 - \|W\|_{L^2}^2).$$

Assume  $4 \le n \le 9$ . Then (2/n) - ((n-2)/4(n-1)) > 0 so that

(11) 
$$C_n^{-1} \|W\|_{L^{n/2}} \ge \frac{n-2}{4(n-1)} s.$$

If, contrarily,  $n \geq 10$ , it holds (2/n) - (n-2/4(n-1)) < 0. However  $\|W\|_{L^2}^2 \leq \|W\|_{L^p}^2$ , since p > 2. So,

(12) 
$$\left(\frac{2}{n} - \frac{n-2}{4(n-1)}\right) s \|W\|_{L^{2}}^{2}$$

$$\geq \left(\frac{2}{n} - \frac{n-2}{4(n-1)}\right) s \|W\|_{L^{p}}^{2}.$$

We have thus

(13) 
$$C_n^{-1} \|W\|_{L^{n/2}} \|W\|_{L^p}^2$$
  

$$\geq \left\{ \left( \frac{2}{n} - \frac{n-2}{4(n-1)} \right) s + \frac{n-2}{4(n-1)} s \right\} \|W\|_{L^p}^2$$

$$= \frac{2}{n} s \|W\|_{L^p}^2$$

giving rise to  $||W||_{L^{n/2}} \geq (2/n)C_n s$ .

Therefore, if we put C(n) as

(14) 
$$C(n) = \frac{n-2}{4(n-1)}C_n, \quad 4 \le n \le 9,$$

$$(15) \qquad \qquad = \frac{2}{n}C_n, \qquad 10 \le n,$$

then we get a contradiction giving the complete proof.

**4. Remarks.** A pointwise isolation theorem is similarly obtained as

**Theorem 0.4.** Let (M,g) be a compact connected oriented Einstein n-manifold  $(n \geq 4)$  of scalar curvature s > 0. If  $|W| \leq (2/n)C_ns$  holds everywhere and strict inequality holds at a point, then W = 0, that is, (M,g) is, up to a constant scale, a finite isometric quotient of the standard n-sphere.

In four dimension we have a similar pointwise isolation theorem for the (anti-)self-dual Weyl tensor  $W^{\pm}$  for a wide class of 4-manifolds satisfying  $\delta^{\pm}W^{\pm}=0$  in [6]. Remark also that M. Gursky showed the following in [3]:

Let M be a compact, connected, oriented 4-manifold and g a metric on M which satisfies  $\delta^+W^+=0$  and the Yamabe constant  $\mu([g])>0$ . If

$$\int_{M} |W^{+}|^{2} dv_{g} \leq \frac{4}{3} \pi^{2} (2\chi(M) + 3\tau(M)),$$

then, (i) g is anti-self-dual, i.e.,  $W^+=0$ , or (ii) g is a positive Einstein metric which is either Kähler or the quotient of a Kähler metric by an isometric anti-holomorphic involution.

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