Ichimura-Sumida criterion for Iwasawa λ-invariants

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Abstract: For an odd prime number p and real abelian number fields k with the degree $[k:\mathbf{Q}]=p$ in which p splits completely, we give a criterion for vanishing of Iwasawa λ-invariants.

Key words: Iwasawa invariant; cyclotomic unit; cubic field.

1. Introduction. Let p be an odd prime number, k a real abelian number field, k_∞ the cyclotomic $\mathbf{Z}_p$-extension of k. Greenberg's conjecture asserts that the Iwasawa λ-invariant $\lambda_p(k)$ of k_∞ over k vanishes. In [3], [5] and [6], remarkable criteria of $\lambda_p(k)=0$ were established when the degree $[k:\mathbf{Q}]$ is prime to p. In this paper we deal with the case $[k:\mathbf{Q}]=p$.

All the auhors of [3], [5] and [6] had the same idea to obtain their criteria by investigating deeply the properties of cyclotomic units. But the formulations of their criteria are not the same, because of their different approaches. For example, in [3], Ichimura and Sumida gave a criterion by using a structure theorem of semi-local units modulo cyclotomic units proved in [2]. When p splits completely and $[k:\mathbf{Q}]=p$, we shall give a criterion similar to [3] by using Tsuji's result in [9] in place of [2], which will be applied to cyclic cubic fields to give new examples of $\lambda_3(k)=0$.

2. Theorem. We begin by explaining the notations. We denote as usual by $\mathbf{Z}$ and $\mathbf{Q}$ the ring of rational integers and the field of the rational numbers, respectively.

For a positive integer n, we denote by ζ_n a primitive n-th root of unity. Let p be a fixed odd prime number, $\overline{\mathbf{Q}}_p$ the algebraic closure of the p-adic number field $\mathbf{Q}_p$ and $\mathfrak{O}=\mathbf{Z}_p[\zeta_p]$ the integer ring of $\mathbf{Q}_p(\zeta_p)$. For a finite extension L over K, we denote by $N_{L/K}$ the norm mapping of L over K.

For any algebraic number field F, we denote respectively by $\lambda_p(F)$ and $\mu_p(F)$ the Iwasawa λ-invariant and μ-invariant associated to the ideal class group of the cyclotomic $\mathbf{Z}_p$-extension $F_\infty=\cup_{n=0}^{\infty}F_n$ over F with its n-th layer F_n.

Let k be a cyclic extension of $\mathbf{Q}$ of degree p with conductor f and $\Delta=G(k_\infty/\mathbf{Q}_\infty)$ the Galois group of k_∞ over $\mathbf{Q}_\infty$. We fix a topological generator γ of $G(k_\infty/k)$. In this paper, we shall always assume that the prime number p splits completely in k.

Let χ be a non-trivial $\overline{\mathbf{Q}}_p$-valued character of Δ, $g_\chi(T)$ the Iwasawa power series in the power series ring $\mathfrak{O}[[T]]$ associated to the p-adic L-function $L_p(s,\chi)$ by

$$g_\chi((1+fp)^{1-s}-1)=L_p(s,\chi)$$

for $s\in\mathbf{Z}_p$ and $P_\chi(T)$ the distinguished polynomial of $g_\chi(T)$. We decompose

$$P_\chi(T)=P_1(T)^{e_1}\cdots P_r(T)^{e_r}$$

for some $r\geq 1$ and some natural numbers e_i, where $P_i(T)$'s are some irreducible distinguished polynomials in $\mathfrak{O}[T]$ with $P_i\neq P_j \quad (i\neq j)$. We note that the Iwasawa main conjecture and Kida's formula imply $r\geq 1$.

Put $\nu_n=\nu_n(T)=((1+T)^{p^n}-1)/T$ for $n\geq 1$. By the Leopoldt conjecture and Iwasawa main conjecture, which have already been proved in our case, the abelian group $\mathfrak{O}[[T]]/(P_i,\nu_n)$ is finite. We denote by $p^{a_{i,n}}$ the exponent of $\mathfrak{O}[[T]]/(P_i,\nu_n)$. Then we can take a polynomial $X_{P_i,n}(T)=X_{i,n}(T)$ in $\mathfrak{O}[T]$ satisfying

$$X_{i,n}P_i\equiv p^{a_{i,n}} \pmod{\nu_n}$$

for $n\geq 1$. Define a polynomial $Y_{i,n}(S,T)$ in $\mathbf{Z}[S,T]$ by

$$Y_{i,n}(\chi(\sigma),T)\equiv X_{i,n}(T) \pmod{p^{a_{i,n}}},$$

where σ is a fixed generator of Δ.

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Let $\mathfrak{p}_n$ be the unique prime ideal of $\mathbf{Q}_n$ lying above p, E_{k_n} the group of units ε of k_n satisfying $\varepsilon \equiv 1 \pmod{\mathfrak{p}_n}$ and $C_{\mathbf{Q}_n}$ the group of cyclotomic units ε of $\mathbf{Q}_n$ satisfying $\varepsilon \equiv 1 \pmod{\mathfrak{p}_n}$. Put

$$c_n = N_{\mathbf{Q}(\zeta_{fp^{n+1}})/k_n}(1-\zeta_{fp^{n+1}})^{p-1}.$$

Then c_n is contained in E_{k_n}. Our main purpose is to prove the following theorem and to give new examples of cyclic cubic fields k's for which $\lambda_3(k)=0$.

Theorem 2.1. *Let p be an odd prime number and k a cyclic extension of $\mathbf{Q}$ of degree p in which p splits completely. Let $\gamma, \zeta_n, Y_{i,n}, E_{k_n}, C_{\mathbf{Q}_n}, p^{a_{i,n}}, \sigma$ be as above. Then the following are equivalent:*

(1) $\lambda_p(k)=0$.

(2) *There exists $n \geq 1$ such that there exists no element ε_n of E_{k_n} with*

$$\varepsilon_n^{p^{a_{i,n}}} \equiv c_n^{Y_{i,n}(\sigma,\gamma-1)} \pmod{C_{\mathbf{Q}_n}}$$

for any i $(1 \leq i \leq r)$.

Corollary 2.1. *Assumptions and notations being as above, we have $\lambda_p(k)=0$, if there exists $n \geq 1$ such that*

$$c_n^{Y_{i,n}(\sigma,\gamma-1)(\sigma-1)} \notin (E_{k_n})^{p^{a_{i,n}}}$$

for any i $(1 \leq i \leq r)$.

3. Proof of Theorem. Now we consider several $\mathfrak{O}[[T]]$-modules in order to prove the theorem. Let $\mathbf{Z}_p[\chi]$ denote a free $\mathfrak{O}$-module of rank one on which Δ acts via χ. For any $\mathbf{Z}_p[\Delta]$-module $\mathfrak{M}$, we define the following $\mathfrak{O}$-module; $\mathfrak{M}(\chi) = \mathfrak{M} \otimes_{\mathbf{Z}_p[\Delta]} \mathbf{Z}_p[\chi]$. Moreover we put $N_\Delta = \sum_{\sigma\in\Delta}\sigma$ and $N_\Delta\mathfrak{M} = \{\sum_{\sigma\in\Delta}\sigma m \mid m \in \mathfrak{M}\}$. Then the following is well-known.

Lemma 3.1 (cf. [5]). *The $\mathfrak{O}$-module $\mathfrak{M}(\chi)$ is isomorphic to $\mathfrak{M}/N_\Delta\mathfrak{M}$ as $\mathbf{Z}_p[\Delta]$-module.*

Let $L(k_\infty)$ be the maximal unramified abelian p-extension of k_∞ and $M(k_\infty)$ the maximal abelian p-extension of k_∞ unramified outside p. Moreover we put $\Lambda = \mathbf{Z}_p[[T]]$, $X = G(L(k_\infty)/k_\infty)$, $\mathfrak{X} = G(M(k_\infty)/k_\infty)$. Then we regard X and $\mathfrak{X}$ as $\Lambda[\Delta]$-modules, where $1+T$ acts as the fixed topological generator γ of $G(k_\infty/k)$. Then we have the following:

Lemma 3.2 (cf. [5]). *We have $N_\Delta X = 0$ and $N_\Delta\mathfrak{X}=0$, which implies $X(\chi)\cong X$ and $\mathfrak{X}(\chi)\cong\mathfrak{X}$.*

Lemma 3.3. *The λ-invariant $\lambda_p(k)$ is $(p-1)$ times the free $\mathfrak{O}$-rank of $X(\chi)$.*

Now we denote by $U_{\mathfrak{p}_n}$ the principal units of the completion $\mathbf{Q}_{n\mathfrak{p}_n}$ of $\mathbf{Q}_n$ at the prime $\mathfrak{p}_n$, by $E_{\mathfrak{p}_n}$ the closure of $E_{\mathbf{Q}_n}$ in $U_{\mathfrak{p}_n}$ and by $C_{\mathfrak{p}_n}$ the closure of $C_{\mathbf{Q}_n}$ in $U_{\mathfrak{p}_n}$. Put

$$V_{\mathfrak{p}_n} = \{u \in U_{\mathfrak{p}_n} \mid N_{\mathbf{Q}_{n\mathfrak{p}_n}/\mathbf{Q}_p}(u)=1\} = \bigcap_{m>n} N_{\mathbf{Q}_{m\mathfrak{p}_m}/\mathbf{Q}_{n\mathfrak{p}_n}}(U_{\mathfrak{p}_m})$$

(cf. [10, p. 310]). Then we have the following:

Lemma 3.4. *The assumptions and notations being as above, we have $E_{\mathfrak{p}_n} = C_{\mathfrak{p}_n} = V_{\mathfrak{p}_n}$.*

Proof. Since the class number of $\mathbf{Q}_n$ is prime to p, the group index $(E_{\mathbf{Q}_n} : C_{\mathbf{Q}_n})$ is also prime to p. This shows $E_{\mathfrak{p}_n} = C_{\mathfrak{p}_n}$. Moreover the norm mapping is continuous, which shows $E_{\mathfrak{p}_n} \subset V_{\mathfrak{p}_n}$. Now we assume $E_{\mathfrak{p}_n} \subsetneqq V_{\mathfrak{p}_n}$. Since $M(\mathbf{Q}_n) = \mathbf{Q}_\infty$ implies $U_{\mathfrak{p}_n}/E_{\mathfrak{p}_n} \cong \mathbf{Z}_p$, the index $(U_{\mathfrak{p}_n} : V_{\mathfrak{p}_n})$ is finite. Put $(U_{\mathfrak{p}_n} : V_{\mathfrak{p}_n}) = p^a$, then $(1+p)^{p^a} \in V_{\mathfrak{p}_n}$. This is a contradiction. □

Let $\mathfrak{P}_n$ be a prime ideal of k_n lying above $\mathfrak{p}_n$ and $U_{\mathfrak{P}_n}$ the principal units of $k_{n\mathfrak{P}_n}$. We put

$$U_n = \prod_{i=1}^{p} U_{\mathfrak{P}_n^{\sigma^i}} \quad\text{and}\quad V_n = \prod_{i=1}^{p} V_{\mathfrak{P}_n^{\sigma^i}},$$

where

$$V_{\mathfrak{P}_n} = \{u \in U_{\mathfrak{P}_n} \mid N_{k_{n\mathfrak{P}_n}/\mathbf{Q}_p}(u)=1\}.$$

Then $\Lambda[\Delta]$ acts on U_n in the obvious way. We fix an isomorphism φ of $\mathbf{Q}_{n\mathfrak{p}_n}$ onto $k_{n\mathfrak{P}_n}$. Then φ induces an isomorphism $\widetilde{\varphi} : U_{\mathfrak{p}_n} \otimes_{\mathbf{Z}_p} \mathbf{Z}_p[\Delta] \to U_n$ as $\Lambda[\Delta]$-module in the obvious way.

Let $\iota : E_{k_n} \to U_n$ be the diagonal embedding. We note $\varepsilon^\iota = (\sum_{i=1}^p \varepsilon^{\sigma^{-i}} \otimes \sigma^i)^{\widetilde{\varphi}}$ for $\varepsilon \in E_{k_n}$. We identify U_n with $U_{\mathfrak{p}_n} \otimes_{\mathbf{Z}_p} \mathbf{Z}_p[\Delta]$ by $\widetilde{\varphi}$. Now, we determine the structure of $V_n(\chi)$ as an $\mathfrak{O}[[T]]$-module. Since $V_{\mathfrak{p}_n} = C_{\mathfrak{p}_n}$ is generated by

$$N_{\mathbf{Q}(\zeta_{p^{n+1}})/\mathbf{Q}_n}\left(\frac{1-\zeta_{p^{n+1}}^g}{1-\zeta_{p^{n+1}}}\right)^{p-1}$$

as Λ-module, $V_{\mathfrak{p}_n}$ is isomorphic to $\Lambda/\nu_n\Lambda$, where g is a primitive root modulo p^{n+1}. Hence we have

$$V_n \cong \Lambda[\Delta]/\nu_n\Lambda[\Delta], \tag{1}$$

which shows

$$\begin{aligned} (2)\quad V_n &\cong (\Lambda[\Delta]/\nu_n\Lambda[\Delta])/((N_\Delta\Lambda[\Delta]+\nu_n\Lambda[\Delta])/\nu_n\Lambda[\Delta]) \\ &\cong \Lambda[\Delta]/(N_\Delta\Lambda[\Delta]+\nu_n\Lambda[\Delta]) \\ &\cong (\Lambda[\Delta]/N_\Delta\Lambda[\Delta])/((N_\Delta\Lambda[\Delta]+\nu_n\Lambda[\Delta])/N_\Delta\Lambda[\Delta]) \\ &\cong \mathfrak{O}[[T]]/\nu_n\mathfrak{O}[[T]]. \end{aligned}$$

Especially we have

$$V_n(\chi) \cong \mathbf{Z}_p^{p^n(p-1)} \tag{3}$$

as $\mathbf{Z}_p$-module.

Let $\mathcal{E}_n$ be the closure of $(E_{k_n})^\iota$ in U_n and $W_n = N_\Delta(V_n)$.

Lemma 3.5. *The above $\mathcal{E}_n$ contains W_n.*

Proof. We note $W_n = \{\sum_{i=1}^p u \otimes \sigma^i \mid u \in V_n\}$. Hence W_n is the closure of $\{\sum_{i=1}^p u^{\sigma^{-i}} \otimes \sigma^i \mid u \in E_{\mathbf{Q}_n}\}$ by Lemma 3.4. This shows $W_n \subset \mathcal{E}_n$. □

Put $U = \varprojlim U_n$, $\mathcal{E} = \varprojlim \mathcal{E}_n$ and $W = \varprojlim W_n$. Here the projective limits are taken with respect to the relative norm. Then we have $U = \varprojlim V_n$ and the projection $U \to V_n$ is surjective by the definition of V_n. Hence we have $U \cong \Lambda[\Delta]$ by (1), $W = N_\Delta U$ and $U(\chi) \cong \mathfrak{O}[[T]]$. Recall $I = G(M(k_\infty)/L(k_\infty))$, which is isomorphic to $U/\mathcal{E}$ by class field theory. Then we have

$$\begin{aligned} I(\chi) &\cong (U/\mathcal{E})(\chi) \cong (U/\mathcal{E})/(\mathcal{E}W/\mathcal{E}) \cong U/\mathcal{E} \cong I \\ &\cong (U/W)/(\mathcal{E}/W) \end{aligned}$$

by Lemma 3.5. Hence we have the following exact sequence of $\mathfrak{O}[[T]]$-modules by Lemma 3.2:

$$0 \to I(\chi) \to \mathfrak{X}(\chi) \to X(\chi) \to 0.$$

This implies $\Phi_{\mathfrak{X}}(T) = \Phi_I(T)\Phi_X(T)$ for the characteristic polynomials $\Phi_I(T)$, $\Phi_{\mathfrak{X}}(T)$, $\Phi_X(T)$ of $\mathfrak{O}[[T]]$-modules $I(\chi)$, $\mathfrak{X}(\chi)$, $X(\chi)$, respectively. Moreover we have $\Phi_{\mathfrak{X}}(T) = P_\chi(T)$ by the Iwasawa main conjecture. Put $Q_i(T) = P_\chi(T)/P_i(T)$. Then we have the following:

Lemma 3.6. *The irreducible polynomial $P_i(T)$ divides $\Phi_X(T)$ if and only if $(V_n/W_n)^{Q_i} \subset \mathcal{E}_n/W_n$ for all $n \geq 1$.*

Proof. We note $V_n(\chi) = V_n/W_n$ in our case. Then the proof is essentially the same as in [5, p. 732]. □

Now, we denote c_n^ι by $\eta_n \in U_n$. Let C_n be the closed subgroup of U_n generated by η_n as $\Lambda[\Delta]$-module and put $C = \varprojlim C_n$. Then C is a closed subgroup of U generated by $\eta = \varprojlim \eta_n$ as $\Lambda[\Delta]$-module. Then we have

$$\begin{aligned} (U/C)(\chi) &\cong (U/W)/(CW/W) \\ &\cong \varprojlim(U_n/W_n)/(C_nW_n/W_n). \end{aligned}$$

Proof of Theorem 2.1. Our proof is essentially the same as in [3, p. 733] except for using [9] and dealing with W_n. By [9], there exists an element $u = \varprojlim u_n$ such that uW generates U/W as $\mathfrak{O}[[T]]$-module, $u^{P_\chi} \equiv \eta \pmod{W}$ and $u_n^{P_\chi} \equiv \eta_n \pmod{W_n}$. Hence

$$C_nW_n/W_n \cong \left(P_\chi \mathfrak{O}[[T]] + \nu_n \mathfrak{O}[[T]]\right) \Big/ \nu_n \mathfrak{O}[[T]]$$

by (2). Hence we have

$$\begin{aligned} \eta_n^{X_{i,n}} &\equiv u_n^{X_{i,n}P_\chi} = u_n^{X_{i,n}P_iQ_i} \\ &\equiv u_n^{p^{a_{i,n}}Q_i} \pmod{W_n} \end{aligned} \tag{4}$$

We suppose that there exists an element $\varepsilon_n \in \mathcal{E}_n$ with $\eta_n^{X_{i,n}} \equiv \varepsilon_n^{p^{a_{i,n}}} \pmod{W_n}$. Then we have $\varepsilon_n^{a_{i,n}} \equiv u_n^{p^{a_{i,n}}Q_i} \pmod{W_n}$ by (4). Hence $\varepsilon_n \equiv u_n^{Q_i} \pmod{W_n}$ by (3). This means $(V_n/W_n)^{Q_i} \subset \mathcal{E}_n/W_n$. Conversely, we suppose that $(V_n/W_n)^{Q_i} \subset \mathcal{E}_n/W_n$. Then there exists an element ε_n of $\mathcal{E}_n$ with $u_n^{Q_i} \equiv \varepsilon_n \pmod{W_n}$, which shows $\eta_n^{X_{i,n}} \equiv \varepsilon_n^{p^{a_{i,n}}} \pmod{W_n}$ by (4). The diagonal mapping $E_{k_n} \to \mathcal{E}_n$ induces an isomorphism $E_{k_n}/E_{k_n}^{p^{a_{i,n}}} \cong \mathcal{E}_n/\mathcal{E}_n^{p^{a_{i,n}}}$ by [10, p. 75] which shows

$$\begin{aligned} &\left(E_{k_n}/E_{\mathbf{Q}_n}\right) \Big/ \left(E_{\mathbf{Q}_n} E_{k_n}^{p^{a_{i,n}}}/E_{\mathbf{Q}_n}\right) \\ &\cong \left(\mathcal{E}_n/W_n\right) \Big/ \left(\mathcal{E}_n^{p^{a_{i,n}}} W_n/W_n\right) \end{aligned}$$

by the proof of Lemma 3.5. Since $(E_{\mathbf{Q}_n} : C_{\mathbf{Q}_n})$ is prime to p, we have

$$\begin{aligned} &\left(E_{k_n}/C_{\mathbf{Q}_n}\right) \Big/ \left(C_{\mathbf{Q}_n} E_{k_n}^{p^{a_{i,n}}}/C_{\mathbf{Q}_n}\right) \\ &\cong \left(\mathcal{E}_n/W_n\right) \Big/ \left(\mathcal{E}_n^{p^{a_{i,n}}} W_n/W_n\right). \end{aligned}$$

This shows our theorem. □

4. Examples. In [1], we studied λ_3-invariants of cyclic cubic fields k of prime conductor f in which 3 splits. We restricted our attention to f's such that $f \equiv 1 \pmod{3^2}$ and $f \not\equiv 1 \pmod{3^3}$. Our method was based on the explicit construction of the cyclotomic units in k_2 and succeeded in proving $\lambda_3(k) = 0$ for some f's. But there remain three tough f's for which we were not able to determine $\lambda_3(k)$, namely f =5527, 7219 and 8677. In this section, we try to attack these f's.

Since 3^2 is the highest power of 3 dividing $f-1$, we see immediately that $\deg P_\chi(T) = 2$ by Kida's formula (cf. [4]). We have to examine whether $P_\chi(T)$ is irreducible and factorize it when it is not irreducible. We first note that $g_\chi(T)$ is constructed explicitly as follows. Put $\chi^* = \omega\chi^{-1}$ with the Teichmüller character ω modulo 3. Let

$$\xi_n = -\frac{1}{2q_n} \sum_{\substack{a=1 \\ (a,q_n)=1}}^{q_n} a\chi^*(a)^{-1} \left(\frac{\mathbf{Q}_n}{a}\right)^{-1} \in \mathfrak{O}[\Gamma_n]$$

be $-1/2$ times Stickelberger element for $k_n(\zeta_3)$. Here $q_n = 3^{n+1}f$, Γ_n is the Galois group $G(\mathbf{Q}_n/\mathbf{Q})$ and

$(\mathbf{Q}_n/a)$ denotes the Artin symbol. Let $g_n(T)$ be the power series in $\mathfrak{O}[[T]]$ associated to ξ_n via correspondence $1+\dot{T} \leftrightarrow (\mathbf{Q}_n/(1+q_0))$, where $(1+T)(1+\dot{T}) = 1+q_0$. Then $g_\chi(T) = \varprojlim g_n(T)$ and $g_\chi(T)$ is explicitly approximated by g_n:

$$g_\chi(T) \equiv g_n(T) \pmod{\omega_n(\dot{T})}, \tag{5}$$

where $\omega_n = \omega_n(T) = (1+T)^{3^n} - 1$. Let $\pi = 1 - \zeta_3$ be a prime element of $\mathfrak{O} = \mathbf{Z}_3[\zeta_3]$. Since $P_\chi(T)$ is a distinguished polynomial, we have $\alpha \in \pi\mathfrak{O}$, if $P_\chi(T)$ has a root α in $\mathfrak{O}$. The following lemma gives a sufficient condition for reducibility of $P_\chi(T)$.

Lemma 4.1. *Let $r < n$ be positive integers. If there exists a representative α_0 of $\pi\mathfrak{O}/\pi^{2n+1}\mathfrak{O}$ such that*

$$g_n(\alpha_0) \equiv 0 \pmod{\pi^{2n+1}}$$

and

$$g_n'(\alpha_0) \not\equiv 0 \pmod{\pi^{r+1}},$$

(here g_n' denotes the formal derivative of g_n), then $P_\chi(T)$ has a root α in $\pi\mathfrak{O}$. Furthermore,

$$\alpha \equiv \alpha_0 \pmod{\pi^{2n+1-2r}}.$$

Proof. Since $g_\chi(T) = P_\chi(T)u_\chi(T)$ with a unit element $u_\chi(T)$ in $\mathfrak{O}[[T]]$ and $\omega_n(\dot{\alpha_0}) = (1+\dot{\alpha_0})^{3^n} - 1 \equiv 0 \pmod{\pi^{2n+1}}$, we have

$$P_\chi(\alpha_0) \equiv 0 \pmod{\pi^{2n+1}}$$

by (5). Furthermore, since $g_\chi'(T) = P_\chi'(T)u_\chi(T) + P_\chi(T)u_\chi'(T)$, we have

$$P_\chi'(\alpha_0) \not\equiv 0 \pmod{\pi^{r+1}}$$

again by (5). Then we apply Proposition 2 in [7] to our case. □

We note that if we have $\beta_0 \in \pi\mathfrak{O}$ such that

$$g_n(\beta_0)/g_n'(\beta_0)^2 \in \pi\mathfrak{O}$$

then we can easily get α_0 in Lemma 4.1 by the Newton iteration

$$\beta_{i+1} = \beta_i - g_n(\beta_i)/g_n'(\beta_i).$$

Example 4.1. Let $f = 7219$. Then

$$\begin{aligned} g_4(T) \equiv{}& (27+153\zeta) + (98+145\zeta)T + (31+181\zeta)T^2 \\ &+(160+225\zeta)T^3 + (87+140\zeta)T^4 \\ &+(231+151\zeta)T^5 + (234+86\zeta)T^6 \\ &+(12+125\zeta)T^7 + (107+184\zeta)T^8 \\ &+ \text{higher terms} \pmod{3^5}, \end{aligned}$$

where $\zeta = \zeta_3$. By Lemma 4.1, we see that $P_\chi(T)$ decomposes into a product of linear factors $P_1(T)P_2(T)$, where

$$\begin{aligned} P_1(T) &\equiv T - (12+42\zeta) \pmod{\pi^7}, \\ P_2(T) &\equiv T - (50+16\zeta) \pmod{\pi^7}. \end{aligned}$$

Then $a_{1,2} = 2$ and $a_{2,2} = 3$. Namely the exponents of $\mathfrak{O}[[T]]/(P_1, \nu_2)$ and $\mathfrak{O}[[T]]/(P_2, \nu_2)$ are 3^2 and 3^3 respectively. So it is enough to determine $X_{i,2}(T)$ modulo 3^2 or 3^3. In fact, we have

$$\begin{aligned} X_{1,2}(T) \equiv{}& 3T + 3T^4 + (3+6\zeta)T^6 + T^7 \pmod{3^2}, \\ X_{2,2}(T) \equiv{}& (24+21\zeta) + (21+9\zeta)T + 9\zeta T^2 \\ &+(6+3\zeta)T^3 + (24+15\zeta)T^4 + (3+3\zeta)T^5 \\ &+(5+16\zeta)T^6 + T^7 \pmod{3^3} \end{aligned}$$

and hence

$$\begin{aligned} Y_{1,2}(\sigma,\gamma-1) \equiv{}& (2+6\sigma) + 7\gamma + 6\gamma^2 + (8+6\sigma)\gamma^3 \\ &+4\gamma^4 + 3\gamma^5 + (5+6\sigma)\gamma^6 \\ &+\gamma^7 \pmod{3^2}, \\ Y_{2,2}(\sigma,\gamma-1) \equiv{}& (22+19\sigma) + (16+21\sigma)\gamma \\ &+(15+3\sigma)\gamma^2 + (10+4\sigma)\gamma^3 \\ &+(22+24\sigma)\gamma^4 + (21+15\sigma)\gamma^5 \\ &+(25+16\sigma)\gamma^6 + \gamma^7 \pmod{3^3}. \end{aligned}$$

Now it is a routine work to check whether an integer of k_2 is an odd power in k_2. We see that $c_2^{Y_{1,2}(\sigma,\gamma-1)(\sigma-1)}$ is cube but not 9-th power in k_2 and $c_2^{Y_{2,2}(\sigma,\gamma-1)(\sigma-1)}$ is also cube but not 9-th power in k_2. Hence we can conclude that $\lambda_3(k) = 0$ by Corollary 2.1.

Example 4.2. Let $f = 8677$. In a similar manner to Example 4.1, we have $P_\chi(T) = P_1(T)P_2(T)$, where

$$\begin{aligned} P_1(T) &\equiv T - (15+174\zeta) \pmod{\pi^9}, \\ P_2(T) &\equiv T - (197+151\zeta) \pmod{\pi^9}. \end{aligned}$$

We have $a_{1,2} = 2$ and $a_{2,2} = 3$. But $c_2^{Y_{1,2}(\sigma,\gamma-1)(\sigma-1)}$ is 9-th power in k_2. So we have to work in k_3. In this case, we have $a_{1,3} = 3$ and $a_{2,3} = 4$. Fortunately, we see that $c_3^{Y_{1,3}(\sigma,\gamma-1)(\sigma-1)}$ is 9-th power but not 27-th power in k_3 and $c_3^{Y_{2,3}(\sigma,\gamma-1)(\sigma-1)}$ is cube but not 9-th power in k_3. Hence we can conclude that $\lambda_3(k) = 0$ by Corollary 2.1.

Next we consider the case that $P_\chi(T)$ does not decompose into a product of linear factors. The following lemma gives a sufficient condition for irreducibility of $P_\chi(T)$ when $\deg P_\chi(T) = 2$.

Lemma 4.2. *Let n be a positive integer. If $g_n(x) \not\equiv 0 \pmod{\pi^{2n+1}}$ for any representative x of $\pi\mathfrak{O}/\pi^{2n+1}\mathfrak{O}$, then $P_\chi(T)$ has no roots in $\mathfrak{O}$.*

Proof. Let $g_\chi(T) = P_\chi(T)u_\chi(T)$ with a unit element $u_\chi(T)$ in $\mathfrak{O}[[T]]$. If $P_\chi(x) = 0$ for some $x \in \mathfrak{O}$, then $x \in \pi\mathfrak{O}$ and $g_\chi(x) = 0$. We have $g_n(x) \equiv 0 \pmod{\pi^{2n+1}}$ from (5). □

When $P_\chi(T)$ has a irreducible factor $P_i(T)$ of degree greater than one, we approximate $P_\chi(T)$ by $g_n(T)$ using the following lemma which can be proved in the same way as Lemma 5 in [3].

Lemma 4.3. *Assume that* $\deg P_\chi(T) = 2$. *Let τ be a shift operator on $\mathfrak{O}[[T]]$ defined by*

$$\tau\Big(\sum_{i=0}^{\infty} a_i T^i\Big) = \sum_{i=2}^{\infty} a_i T^{i-2}$$

and let $g_n(T) = \pi A_n(T) + T^2 B_n(T)$ with $B_n = \tau(g_n)$. Then

$$P_\chi(T) \equiv \frac{g_n(T)}{B_n(T)} \sum_{j=0}^{\infty} (-1)^j \pi^j \Big(\tau \circ \frac{A_n}{B_n}\Big)^j \circ 1 \pmod{3^n}$$

for $n \geq 2$.

Here, the definition of the operator $\circ$ is the same as in the proof of Proposition 7.2 in [10].

Example 4.3. Let $f = 5527$. Then

$$\begin{aligned} g_2(T) = {} & 6T + 19T^2 + (8+6\zeta)T^3 + 7\zeta T^4 \\ & +(5+2\zeta)T^5 + (7+7\zeta)T^6 + (7+2\zeta)T^7 \\ & +(6+\zeta)T^8 + \text{ higher terms } \pmod{3^2}. \end{aligned}$$

We see that $P_\chi(T) = P_1(T)$ is irreducible by Lemma 4.2 and obtain

$$P_\chi(T) \equiv 6T + T^2 \pmod{3^2}$$

by Lemma 4.3. From this, we see that the exponent of $\mathfrak{O}[[T]]/(P_1, \nu_1)$ is 3 and

$$X_{1,1}(T) \equiv 1 + T \pmod{3}$$

and hence

$$Y_{1,1}(\sigma, \gamma - 1) \equiv \gamma \pmod{3}.$$

We see that $c_1^{Y_{1,1}(\sigma,\gamma-1)(\sigma-1)}$ is not cube in k_1. Hence we can conclude that $\lambda_3(k) = 0$ by Corollary 2.1.

Example 4.4. In [1], we have verified that $\lambda_3(k) = 0$ for seven f's. We can also apply Corollary 2.1 to those f's. We verified that $\lambda_3(k) = 0$ by calculations in k_1 for f =4933, 9001, 9901 and by those in k_2 for f =3907, 6247, 7687, 8011.

For the case f =2269 and 6481, which could not be treated in [1] because $f \equiv 1 \pmod{3^4}$, Ozaki and Yamamoto showed $\lambda_3(k) = 0$ in [8]. Therefore we can conclude that $\lambda_3(k) = 0$ for all cyclic cubic fields of prime conductor less than 10000.

Our method is applicable to k of non-prime conductor. But the factorization of $P_\chi(T)$ becomes difficult along with the growth of $\deg P_\chi(T)$.

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