Exit probability of two-dimensional random walk from the quadrant

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1. Introduction and preliminaries. Let

$$Z_0 = 0, Z_1 = (X_1, Y_1), Z_2 = (X_2, Y_2), ...$$

be a random walk in the two-dimensional integer lattice $\boldsymbol{Z}^2$. By a random walk we mean a stochastic sequence with stationary independent increments starting at the origin. Throughout the paper we impose on the random walk the following assumptions.

Assumption 1.1. *For every* $\boldsymbol{\theta} = (\theta_1, \theta_2)$ *in* $\boldsymbol{R}^2$,

$$\lambda(\boldsymbol{\theta}) := E(e^{\boldsymbol{\theta}\cdot Z_1}) < \infty,$$

where $\boldsymbol{\theta} \cdot \boldsymbol{z}$ *denotes the inner product in* $\boldsymbol{R}^2$.

Let D_i $(i = 1, 2, 3, 4)$ be the *i th quadrant* in $\boldsymbol{R}^2$, that is,

$$D_1 = \{(x, y) \in \boldsymbol{R}^2 | x > 0, y > 0\},$$
$$D_2 = \{(x, y) \in \boldsymbol{R}^2 | x < 0, y > 0\},$$
$$D_3 = \{(x, y) \in \boldsymbol{R}^2 | x < 0, y < 0\},$$

and

$$D_4 = \{(x, y) \in \boldsymbol{R}^2 | x > 0, y < 0\}.$$

Assumption 1.2. $\boldsymbol{\mu} = E(Z_1) \in D_1$, *and* $P(Z_n \in D_4) > 0$ *for some positive integer n.*

Assumption 1.3. *The y-coordinate of the random walk is left-continuous, that is,* $P(Y_1 \in \{-1, 0, 1, 2, ...\}) = 1$.

Let a and b be positive integers. In this paper we will take a arbitrarily fixed, so we omit a in many of our statements and notations. Set

$$T_b := \inf\{n \geq 0 | (a, b) + Z_n \notin D_1\}$$

$(\inf \emptyset = \infty)$. Define

$$D_4^* := \{(x, y) | x > 0,\ y \leq 0\}$$

and

$$r_b := P(T_b < \infty, (a, b) + Z_{T_b} \in D_4^*).$$

Since $Z_n \sim \boldsymbol{\mu} n$ *a.s.* $(n \to \infty)$ by the *strong law of large numbers*, we have $r_b \to 0$ $(b \to \infty)$ from the first condition of Assumption 1.2. The purpose of this paper is to study the decay rate of r_b to 0. Our problem is a two-dimensional extension of the asymptotic analysis of *ruin probability* for one dimensional random walk with positive drift.

Let Θ denote the contour of the moment generating function $\lambda(\boldsymbol{\theta})$ at the level 1, that is, $\Theta = \{\boldsymbol{\theta} \in \boldsymbol{R}^2 | \lambda(\boldsymbol{\theta}) = 1\}$. It is shown from Assumptions 1.1 and 1.2 the following lemma. (See, e.g., Ney *et al.* [4]).

Lemma 1.1. *Θ is a smooth convex curve. Moreover, it intersects the θ_2-axis at two points; the one is the origin and the other is* $\widetilde{\boldsymbol{\theta}} = (0, \widetilde{\theta}_2)$ *with* $\widetilde{\theta}_2 < 0$.

Note that, if $\boldsymbol{\theta} \in \Theta$, *then* $\exp(\boldsymbol{\theta} \cdot \boldsymbol{z})$ *is a harmonic function of the random walk, namely, it satisfies*

$$E(\exp\{\boldsymbol{\theta} \cdot (Z_1 + \boldsymbol{z})\}) = \exp(\boldsymbol{\theta} \cdot \boldsymbol{z}) \quad \textit{for all } \boldsymbol{z} \in \boldsymbol{R}^2.$$

From now on we always take $\boldsymbol{\theta}$ as an element of Θ. We will not indicate it in our statements. Let $F(\boldsymbol{z}) := P(Z_1 = \boldsymbol{z})$ and introduce a new probability function on $\boldsymbol{Z}^2$ by

$$F^{(\boldsymbol{\theta})}(\boldsymbol{z}) := \exp(\boldsymbol{\theta} \cdot z) F(\boldsymbol{z}).$$

By $P^{(\boldsymbol{\theta})}$ we denote the probability measure of the random walk with the one-step probability function $F^{(\boldsymbol{\theta})}(\boldsymbol{z})$. By elementary observation we get the following formulas and lemma:

$$\boldsymbol{\mu}^{(\boldsymbol{\theta})} := E^{(\boldsymbol{\theta})}(Z_1) = \nabla \lambda(\boldsymbol{\theta}). \tag{1.1}$$

Lemma 1.2. *The following two statements are equivalent*:
(i) $P^{(\boldsymbol{\theta})}(T_b < \infty) = 1$. (ii) $\boldsymbol{\mu}^{(\boldsymbol{\theta})} \notin D_1$.

Put

$$\eta_b(\boldsymbol{\theta}) := 1(T_b < \infty, (a, b) + Z_{T_b} \in D_4^*) \times \exp(-\boldsymbol{\theta} \cdot Z_{T_b}), \tag{1.2}$$

where $1(A)$ is the indicator function of an event A, that is, $1(A) = 1$ if A occurs and $1(A) = 0$ otherwise. Then, as is shown in Lehtonen et al. [2], we have

$$r_b = E^{(\boldsymbol{\theta})}(\eta_b(\boldsymbol{\theta})). \tag{1.3}$$

As will be discussed in §§ 2 and 3, our key observation on the problem is the following: '*To choose the* $\boldsymbol{\theta}$ *from* Θ *which is most preferable to get an asymptotic formula for* r_b $(b \to \infty)$ *via* (1.3)'. The observation is related to the Monte Carlo analysis for the small values of r_b by *Importance Sampling.* See [2].

2. Classification and results. By Lemma 1.1 we have the tangent of the contour Θ at $\widetilde{\boldsymbol{\theta}}$, which we denote by $\widetilde{L}$. We will observe that the asymptotic formulas may take quite different form if the slope of $\widetilde{L}$ (simply say *the slope*) is positive, zero or negative. Before giving our main results we show some examples with positive and nonpositive slopes.

Example 2.1. The following are random walks with the positive slope.
(i) Random walk with mutually independent x- and y-components.
(ii) Random walk with jumps of size (1,0), (-1,0), (0,1) or (0,-1) (nearest neighbour random walk).

Example 2.2. Consider a random walk with jumps of size (1,2),(-1,1) and (0,-1) with positive probabilities p, q and $r=1-p-q$, respectively. Then Assumption 1.2 is equivalent to $p > q$, $3p+2q > 1$ *and* $r > 0$. Let Assumption 1.2 be satisfied. Then, the slope is positive, zero, or negative according as $p - q^2 - (p+q)^2$ is positive, zero, or negative. For example, if we take $p = 0.6,\ q = 0.3,\ r = 0.1$, the slope is negative. Note that this example satisfies Assumption 2.2 given below.

Let us state our main results.

Theorem 2.1. *Consider a random walk with the positive slope. Then the following formula holds.*

$$r_b \sim K_1 \exp(\widetilde{\theta}_2 b)\ (b \to \infty), \tag{2.1}$$

where K_1 *is the positive constant given by* $K_1 = P^{(\widetilde{\boldsymbol{\theta}})}(a + \inf_{n\geq 0} X_n > 0)$.

Next we consider a random walk with the nonpositive slope. Put

$$\underline{\theta}_2 := \inf\{\theta_2 | (\theta_1, \theta_2) \in \Theta\} \geq -\infty.$$

For a random walk with the zero slope, note that $\widetilde{\theta}_2 = \underline{\theta}_2$.

Theorem 2.2. *For a random walk with the zero slope, we have the following formula.*

$$r_b \sim K_2 b^{-1/2} \exp(\underline{\theta}_2 b)\ (b \to \infty), \tag{2.2}$$

where K_2 *is a positive constant depending only on* F *and* a.

To deal with a random walk with the negative slope, we assume the following in addition to Assumptions 1.1 - 1.3.

Assumption 2.1. $\underline{\theta}_2 > -\infty$.

Theorem 2.3. *Consider a random walk with the negative slope which satisfies Assumption* 2.1 *in addition to Assumptions* 1.1 - 1.3. *Then we have the following upper bound*:

$$r_b = O(b^{-3/2} \exp(\underline{\theta}_2 b))\ (b \to \infty). \tag{2.3}$$

Next we consider a lower bound corresponding to (2.3) for the random walk in Example 2.2. Put

$$\nu_b := \inf\{n \geq 1 | Y_n \leq -b\}\ (\inf \emptyset = \infty).$$

We make the following

Assumption 2.2.

$$\underline{\nu} := E^{(\underline{\boldsymbol{\theta}})}(\nu_1) = \exp\{\sum_1^\infty n^{-1} P^{(\underline{\boldsymbol{\theta}})}(Y_n \geq 0)\} < 6.$$

Theorem 2.4. *Consider the random walk in Example* 2.2 *with the negative slope. Assume that it satisfies Assumption* 2.2. *Then we have*

$$b^{-3/2} \exp(\underline{\theta}_2 b) = O(r_b)\ (b \to \infty). \tag{2.4}$$

We obtain the following from Theorems 2.3 and 2.4.

Theorem 2.5. *For the random walk in Theorem* 2.4,

$$r_b \asymp b^{-3/2} \exp(\underline{\theta}_2 b)\ (b \to \infty). \tag{2.5}$$

3. Proofs of theorems. To prove Theorem 2.1, we apply (1.3) by putting $\boldsymbol{\theta} = \widetilde{\boldsymbol{\theta}}$. Then the result follows immediately from (1.1) and from the strong law of large numbers.

Write $P^{(\underline{\boldsymbol{\theta}})}$ (resp. $E^{(\underline{\boldsymbol{\theta}})}$) as $\underline{P}$ (resp. $\underline{E}$) for simplicity. Consider *the decreasing ladder walk*

$$\widehat{Z}_n = (\widehat{X}_n, \widehat{Y}_n) := Z_{\nu_n}\ (n = 0, 1, 2, ...).$$

(Note that $\widehat{Z}_n$ is defined $\underline{P}$ a.s.. Indeed, $\underline{E}(Y_1) < 0$ implies $\nu_n < \infty$ $\underline{P}$ a.s..) Put

$$\varphi(\theta) := \underline{E}(e^{\theta X_1}),\ \psi(\theta) := \underline{E}(e^{\theta Y_1}),$$

$$\widehat{\varphi}(\theta) := \underline{E}(e^{\theta \widehat{X}_1}) \text{ and } \upsilon(\theta) := \underline{E}(e^{\theta \nu_1})$$

($\theta \in \boldsymbol{R}$). We need the following lemma.

Lemma 3.1. *The following four statements hold.*
(i) *Let* $c := \min\{\psi(\theta),\ \theta \in \mathrm{R}\}$. *Then* $0 < c < 1$, *and the equation* $\varphi(2\theta) = c^{-1}$ *has the unique positive root* d_+ *and the unique negative root* d_-.
(ii) $\widehat{\varphi}(\theta)$ *is finite on the interval* (d_-, d_+), *and the following identity holds.*

(3.1) $\widehat{\varphi}(\theta) = (\varphi(\theta) - 1) \times$

$$\exp\{\sum_{k=1}^{\infty} k^{-1}\underline{E}(1(Y_k \geq 0)\exp(\theta X_k))\} + 1.$$

(iii) *$\widehat{E}(|\widehat{X}_1|^n) < \infty$ for all $n \geq 1$. Especially, $\underline{E}(\widehat{X}_1) = 0$.*
(iv) *$\upsilon(\theta)$ is finite for $\theta < -\log c$, and satisfies*

$$\upsilon(\theta) = 1 - (1-e^{\theta})\exp\{\sum_{k=1}^{\infty} k^{-1}e^{k\theta}\underline{P}(Y_k \geq 0)\}. \tag{3.2}$$

The identities (3.1) and (3.2) follow from the (half-plain) factorization identity. (Spitzer [9] and Mogul'skii et al. [3]. See also Shimura [7].) The proofs of the remaining assertions are elementary.

Proof of Theorem 2.2. By (1.3) we have

$$r_b = \exp(\underline{\theta}_2 b)\underline{E}(1(T_b < \infty, (a,b) + Z_{T_b} \in D_4^*)\exp(-\underline{\theta}_1 X_{T_b})). \tag{3.3}$$

Let

$$\rho_a := \inf\{n \geq 1|\ a + X_n \leq 0\}.$$

for $a \geq 0$. Since $\underline{\theta}_1 = 0$, we have

$$r_b = \exp(\underline{\theta}_2 b)\underline{P}(\rho_a > \nu_b).$$

We get from (3.2) a large deviation type estimate on the distribution of ν_b to yield the following:

$$r_b \geq \exp(\underline{\theta}_2 b)\{\underline{P}(\rho_a > (\underline{\nu} + \delta)b) + O(e^{-\kappa b}))\}$$

and

$$r_b \leq \exp(\underline{\theta}_2 b)\{\underline{P}(\rho_a > (\underline{\nu} - \delta)b) + O(e^{-\kappa b})\}$$

for every positive δ, where κ is a positive constant which may depend on δ. Hence the formula (2.2) follows from the well-known formula $\underline{P}(\rho_a > b) \sim K_3 b^{-1/2}$ $(b \to \infty)$, where K_3 is a positive constant depending only on F and a.

Outline of the Proof of Theorem 2.3. Note that

$$\underline{E}(1(T_b < \infty,\ (a,b) + Z_{T_b} \in D_4^*)\exp(-\theta X_{T_b})) \leq \underline{E}(1(\widehat{\rho}_a > b)\exp(-\theta\widehat{X}_b)),$$

where $\widehat{\rho}_a := \inf\{n \geq 1|\ a + \widehat{X}_n \leq 0\}$. Therefore, Theorem 2.3 follows from the following lemma.

Lemma 3.2. *Let $\theta > 0$. Then we have $\underline{E}(1(\widehat{\rho}_a > b)\exp(-\theta\widehat{X}_b)) \asymp b^{-3/2}$ as $b \to \infty$.*

Outline of Proof of Lemma 3.2. We permute the increments of the random walk to obtain the following:

$$\underline{E}(1(\widehat{\rho}_a > b)\exp(-\theta\widehat{X}_b)) \asymp \sum_{k=0}^{b}\underline{P}(\max\{\widehat{X}_j, 1 \leq j \leq k\} < 0, \widehat{X}_k > -a)\underline{E}(1(\widehat{\rho}_0 > b-k)\exp(-\theta\widehat{X}_{b-k})). \tag{3.4}$$

As is shown in Shimura [6], we have

$$\underline{P}(\max\{\widehat{X}_j, 1 \leq j \leq k\} < 0, \widehat{X}_k > -a) \asymp k^{-3/2}\ (k \to \infty). \tag{3.5}$$

We apply a Tauberian argument to one of the factorization identities (Spitzer [9]) to get

$$\underline{E}(1(\widehat{\rho}_0 > k)\exp(-\theta\widehat{X}_k)) \asymp k^{-3/2} \tag{3.6}$$

$(k \to \infty)$. Putting (3.5) and (3.6) on the right-hand side of (3.4) together, we conclude the desired assertion.

To prove Theorem 2.4 we show the following lemma.

Lemma 3.3. *As $b \to \infty$ we have*

$$b^{-3/2} = O(\underline{P}(\nu_b < \rho_1,\ \widehat{X}_b = 0))$$

Proof of Lemma 3.3. Take a positive $\delta < \underline{\nu} - 1$. Then

$$\underline{P}(\nu_b < \rho_1,\ \widehat{X}_b = 0) > \sum\nolimits_{n:|n-\underline{\nu}b|\leq\delta b}\underline{P}(\rho_1 > n|\nu_b = n, \widehat{X}_b = 0)\underline{P}(\nu_b = n,\ \widehat{X}_b = 0). \tag{3.7}$$

By the local limit theorem (see, e.g., Ibragimov et al. [1]) we have

$$\sum\nolimits_{n:|n-\underline{\nu}b|\leq\delta b}\underline{P}(\nu_b = n,\ \widehat{X}_b = 0) = \underline{P}(\widehat{X}_b = 0) + O(e^{-\kappa b}) \asymp b^{-1/2}\ (b \to \infty). \tag{3.8}$$

Hence we have the lemma if we show the following: *For every n and b with $|n - \underline{\nu}b| \leq \delta b$*

$$b^{-1} = O(\underline{P}(\rho_1 > n|\nu_b = n,\ \widehat{X}_b = 0)) \tag{3.9}$$

$(b \to \infty)$.

Proof of (3.9). Put $\Gamma_n = \{a,b,c\}^n, n = 1,2,\ldots$. For $\boldsymbol{\gamma} = (\gamma_1,\ldots,\gamma_n) \in \Gamma_n$, $x \in \{a,b,c\}$ set $N_0^x(\boldsymbol{\gamma}) = 0$ and

$$N_k^x(\boldsymbol{\gamma}) = \sharp\{1 \leq j \leq k|\gamma_j = x\}\ (k = 1,\ldots,n),$$

where $\sharp A$ denotes the cardinality of a set A. Set

$$\mathcal{X}_k(\boldsymbol{\gamma}) = N_k^a(\boldsymbol{\gamma}) - N_k^b(\boldsymbol{\gamma}),$$
$$\mathcal{Y}_k^0(\boldsymbol{\gamma}) = 2N_k^a(\boldsymbol{\gamma}) + N_k^b(\boldsymbol{\gamma}) - N_k^c(\boldsymbol{\gamma}),$$
$$\mathcal{Y}_k^1(\boldsymbol{\gamma}) = N_k^a(\boldsymbol{\gamma}) + N_k^b(\boldsymbol{\gamma}) - N_k^c(\boldsymbol{\gamma}),$$
$$\mathcal{Y}_k^2(\boldsymbol{\gamma}) = 2N_k^a(\boldsymbol{\gamma}) + 2N_k^b(\boldsymbol{\gamma}) - N_k^c(\boldsymbol{\gamma}),$$

$$\underline{\mathcal{X}}_k(\gamma) = \min_{0\le j\le k} \mathcal{X}_j(\gamma)$$

and

$$\underline{\mathcal{Y}}^i_k(\gamma) = \min_{0\le j\le k} \mathcal{Y}^i_j(\gamma)\ (i = 0, 1, 2).$$

Put

$$\Lambda_{n,b} = \{\gamma \in \Gamma_n | \mathcal{X}_n(\gamma) = 0, \mathcal{Y}^0_n(\gamma) = -b\}$$

and

$$\Lambda^i_{n,b} = \{\gamma \in \Lambda_{n,b} | \underline{\mathcal{Y}}^i_{n-1} > \mathcal{Y}^i_n(\gamma)\}$$

$(i = 0, 1, 2)$. We have

$$\Lambda^2_{n,b} \subseteq \Lambda^0_{n,b} \subseteq \Lambda^1_{n,b},$$

and for $\gamma \in \Lambda_{n,b}$

$$N^a_n(\gamma) = N^b_n(\gamma) = (n-b)/5 \tag{3.10}$$

and

$$N^c_n(\gamma) = (3n + 2b)/5.$$

Let $Q_{n,b}$ and $Q^i_{n,b}$ $(i = 0, 1, 2)$ denote the uniform probability distributions on $\Lambda_{n,b}$ and $\Lambda^i_{n,b}$, respectively. Then we have from (3.10)

$$\begin{aligned} &P(\rho_1 > n | \nu_b = n,\ \widehat{X}_b = 0) = \\ &Q^0_{n,b}(\underline{\mathcal{X}}_n(\gamma) = 0) \ge Q_{n,b}(\Lambda^2_{n,k}) \times \\ &Q_{n,b}(\underline{\mathcal{X}}_n(\gamma) = 0 | \Lambda^2_{n,b}). \end{aligned} \tag{3.11}$$

Let $\lfloor a \rfloor$ denote the integral part of a. We need the following

Lemma 3.4. *Let δ be any fixed positive number. Put $n = \lfloor (1+\delta)b \rfloor$. Then we have*
(i) $Q_{n,b}(\underline{\mathcal{X}}_n = 0 | \Lambda^2_{n,b}) \asymp b^{-1}$ $(b \to \infty)$.
(ii) *Assume further $\delta < 5$. Then*

$$Q_{n,k}(\Lambda^2_{n,b}) \asymp 1\ (b \to \infty).$$

This lemma establishes (3.9). Indeed, we just apply it to the right-hand side of (3.11) by putting $\delta = \underline{\nu} - 1$ (Recall $\delta < 5$ from Assumption 2.2).

Proof of Lemma 3.4. (i) We equip Γ_n with the eqivalence relation $\sim_e$ defined as follows: $\gamma \sim_e \gamma'$ iff

$$N^a_n(\gamma) = N^a_n(\gamma'),\ N^b_n(\gamma) = N^b_n(\gamma'),$$

and

$$\begin{aligned} &\{1 \le j \le n | \gamma_j = a \text{ or } b\} = \\ &\{1 \le j \le n | \gamma'_j = a \text{ or } b\}. \end{aligned}$$

By the local limit theorem

$$\begin{aligned} &\sharp\Lambda^2_{n,b} \asymp \sharp\{\Lambda^2_{n,b}/\sim_e\} \times \\ &(n-b)^{-1/2}\ 2^{2(n-b)/5}\ (n-b \to \infty). \end{aligned} \tag{3.12}$$

Moreover, it follows from the estimate similar to (3.5)

$$\begin{aligned} &\sharp\{\Lambda^2_{n,b} \cap \{\underline{\mathcal{X}}_n = 0\}\} \asymp \\ &\sharp\{\Lambda^2_{n,b}/\sim_e\}(n-b)^{-3/2}\ 2^{2(n-b)/5} \\ &(n-b \to \infty). \end{aligned} \tag{3.13}$$

Hence we have

$$\begin{aligned} &Q_{n,b}(\underline{\mathcal{X}}_n = 0 | \Lambda^2_{n,b}) = \\ &\sharp\{\Lambda^2_{n,b} \cap \{\underline{\mathcal{X}}_n = 0\}\}/\sharp\Lambda^2_{n,b} \asymp \\ &(n-b)^{-1} \asymp b^{-1}\ (b \to \infty). \end{aligned} \tag{3.14}$$

(ii) Put $b' = (6b - n)/5$. Consider the reversed random walk

$$\mathcal{Y}^{2*}_j = \mathcal{Y}^2_{n-j} + b',\ j = 0, 1, ..., n.$$

Since $\mathcal{Y}^2_n(\gamma) = -b'$ for $\gamma \in \Lambda_{n,b}$, with respect to the measure $Q_{n,b}$ $\mathcal{Y}^{2*}_j$, $j = 0, 1, ..., n$, is the pinned random walk which starts from 0 and stops at b' at time n. Note that

$$\Lambda^2_{n,b} = \{\gamma \in \Lambda_{n,b} | \min_{1\le j\le n} \mathcal{Y}^{2*}_j > 0\}$$

and that the mean drift of the pinned random walk $b'/n \sim (5-\delta)/5(1+\delta) > 0$ $(n \to \infty)$. Then we may apply *coupling* (see, e.g., [5]) to show that $Q_{n,k}(\Lambda^2_{n,b})$ is bounded from below by the probability that an appropriately chosen random walk with positive drift never hits $(-\infty, 0]$. Hence we have the desired assertion. (See [8] for more the detail).

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