A Deformation of the Class Number Formula of Real Quadratic Fields

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Abstract: For an odd square-free integer n, there exists a polynomial $L_n(x)$ such that

$$L_n(x) = \sqrt{\phi_n(sx^2)}\exp(-s'\sqrt{n}g_n(x))$$

where $g_n(x) = \sum_{j=0}^{\infty}\left(\frac{n}{2j+1}\right)\frac{x^{2j+1}}{2j+1}$ and $s, s' = \pm 1$.

Using the fact that the value of $g_n(1)$ is related to the class number $h(D)$ of the real quadratic field $\boldsymbol{Q}(\sqrt{n})$ with discriminant D, we deduce a deformation of the class number formula.

First, we set the following notation.

$\phi(n)$ is the Euler function.

$\left(\frac{d}{n}\right)$ is the Jacobi symbol.

$\mu(n)$ is the Möbius function.

$\Phi_n(x)$ is the n-th cyclotomic polynomial.

ζ_n is a primitive n-th root of unity.

(n, k) is the greatest common divisor of k and n.

Let n be a positive odd square-free integer. For given n, we define integers n', s, and s' as follows:

$$n' = \begin{cases} n, & \text{if } n \equiv 1 \bmod 4, \\ 2n, & \text{otherwise.} \end{cases}$$

$$w = \begin{cases} 0, & \text{if } n \text{ has at least two distinct prime factors,} \\ 1, & \text{otherwise.} \end{cases}$$

$$s = \begin{cases} -1, & \text{if } n \equiv 3 \bmod 4, \\ 1, & \text{otherwise.} \end{cases}$$

$$s' = \begin{cases} -1, & \text{if } n \equiv 5 \bmod 8, \\ 1, & \text{otherwise.} \end{cases}$$

If we choose ζ_n to be $e^{\frac{2\pi i}{n}}$, then

$$(1)\quad 2G_n(x) := 2\prod_{\substack{0<j<n \\ (\frac{j}{n})=1}}(x-\zeta_n^j) = A_n(x) - \sqrt{sn}B_n(x),$$

$$(2)\quad 2\tilde{G}_n(x) := 2\prod_{\substack{0<j<n \\ (\frac{j}{n})=-1}}(x-\zeta_n^j) = A_n(x) + \sqrt{sn}B_n(x),$$

where $A_n(x), B_n(x) \in \boldsymbol{Z}[x]$. Note that the particular choice of ζ_n is only significant for the sign of $\sqrt{sn}$ and that $G_n(x)$ (and $\tilde{G}_n(x)$) is symmetric if $n \equiv 1 \bmod 4$ (i.e. if $G_n(x) = a_dx^d + a_{d-1}x^{d-1} + \cdots + a_0$, $a_{d-k} = a_k$ for all k).

We also know that

$$L_n(x) := \prod_{j\in S_n}(x-\zeta_{2n'}^j) = C_n(x^2) - s'x\sqrt{n}D_n(x^2),$$

$$L_n(-x) = \Phi_n(sx^2)/L_n(x) = C_n(x^2) + s'x\sqrt{n}D_n(x^2),$$

where $S_n =$

$$\begin{cases} \{j \mid 0<j<2n',\ (j, n')=1,\ \left(\frac{j}{n}\right) = (-1)^j\}, & \text{if } n \equiv 1 \bmod 4, \\ \{j \mid 0<j<2n',\ (j, n')=1,\ \left(\frac{n}{j}\right) = 1\}, & \text{otherwise.} \end{cases}$$

From the definition, $L_n(x)$ is also symmetric for any n and $C_n(x), D_n(x) \in \boldsymbol{Z}[x]$.

Furthermore, for $|x| \leq 1$, we get the following. (see [1])

$$(3)\quad G_n(x) = \sqrt{\Phi_n(x)}\exp\left(-\frac{s\sqrt{sn}}{2}f_n(x)\right),$$

$$(4)\quad L_n(x) = \sqrt{\Phi_n(sx^2)}\exp(-s'\sqrt{n}g_n(x)),$$

where $f_n(x) = \sum_{j=0}^{\infty}\left(\frac{j}{n}\right)\frac{x^j}{j}$ and

$$g_n(x) = \sum_{j=0}^{\infty}\left(\frac{n}{2j+1}\right)\frac{x^{2j+1}}{2j+1}.$$

As we are interested in the real quadratic fields, suppose $n \equiv 1 \bmod 4$. We want the value of $L_n(1)$. Since $2g_n(x\sqrt{s})/\sqrt{s} = f_n(x) - f_n(-x)$, we need the values of $f_n(1)$ and $f_n(-1)$ in order to get the value of $g_n(1)$.

Step 1. Note that $f_n(1) = L(1, \chi)$ where $\chi(j) = \left(\frac{j}{n}\right)$ is the real, non-trivial Dirichlet character. So the value $f_n(1)$ is related to the class number $h(D)$ of the quadratic field $\boldsymbol{Q}(\sqrt{n})$ with discriminant $D = n'$.

$$f_n(1) = \begin{cases} \dfrac{2\pi}{m\sqrt{n}} h(-n), \\ \quad \text{where } m = \begin{cases} 6, & \text{if } n = 3, \\ 2, & \text{if } n \equiv 3 \bmod 4 \text{ and } n > 3, \end{cases} \\ \dfrac{\ln\varepsilon^2}{\sqrt{n}} h(n), \quad \text{if } n \equiv 1 \bmod 4, \end{cases}$$

where ε is the fundamental unit of $\boldsymbol{Q}(\sqrt{n})$.

Step 2. From (3), $G_n(-1)$ gives the value of $f_n(-1)$. Now we can compute $G_n(-1)$.

Case 1. $n \equiv \pm 1 \bmod 8$.

Since $\left(\dfrac{2}{n}\right) = 1$,

$$G_n(1) = \prod_{(\frac{j}{n})=1} (1 - \zeta_n^{2j}) = G_n(1) \prod_{(\frac{j}{n})=1} (1 + \zeta_n^{j}).$$

Because $G_n(1) \neq 0$, $\Pi_{(\frac{j}{n})=1} (1 + \zeta_n^{j}) = 1$.

$$\therefore \quad G_n(-1) = (-1)^{\frac{\phi(n)}{2}} \prod_{(\frac{j}{n})=1} (1 + \zeta_n^{j}) = 1.$$

Case 2. $n \equiv \pm 3 \bmod 8$.

In this case,

$$\tilde{G}_n(1) := \prod_{(\frac{j}{n})=1} (1 - \zeta_n^{2j}) = G_n(1) \prod_{(\frac{j}{n})=1} (1 + \zeta_n^{j}) = 1.$$

Since $4G_n(1)\tilde{G}_n(1) = 4\Phi_n(1) = 4n^w$,

$$G_n(-1) = (-1)^{\frac{\phi(n)}{2}} \frac{n^w}{G_n(1)^2} = \varepsilon^{2h(n)} \text{ if } n \equiv 5 \bmod 8.$$

Finally, we get

$$\text{(A)} \qquad L_n(1) = \sqrt{\Phi_n(1)}\exp(-s'\sqrt{n}g_n(1)) = \begin{cases} \sqrt{n^w}\varepsilon^{-h(n)}, & \text{if } n \equiv 1 \bmod 8, \\ \sqrt{n^w}\varepsilon^{3h(n)}, & \text{if } n \equiv 5 \bmod 8. \end{cases}$$

In order to get a formula for $L_n(1)$, we need Newton's identities.

Lemma (Newton). *Suppose*

$$P(x) := \prod_{j=1}^{d} (x - \alpha_j) = \sum_{j=0}^{d} a_j x^{d-j},$$

$$P_k := \sum_{i=1}^{d} \alpha_j^k, \quad k = 1, \ldots, d,$$

and the coefficient $a_0 = 1$.

Then

$$ka_k = -\sum_{j=0}^{k-1} P_{k-j}a_j, \quad k = 1, \ldots, d.$$

Now, suppose $L_n(x) = b_0 + b_1 x + \cdots + b_d x^d$ where $d = \deg L_n(x) = \phi(n)$. Since $L_n(x)$ is symmetric for all n,

$$\text{(B)} \qquad L_n(1) = 2\sum_{j=0}^{\phi(n)/2-1} b_j + b_{\frac{\phi(n)}{2}}.$$

Let $g'_k = (k, n')$. The sums Q_k of k-th powers of roots of $L_n(x)$ are

$$\text{(C)} \qquad Q_k = \begin{cases} \left(\dfrac{n}{k}\right) s'\sqrt{n}, & \text{if } k \text{ is odd}, \\ \mu\left(\dfrac{n'}{g'_k}\right)\phi(g'_k)\cos\left(\dfrac{(n-1)k\pi}{4}\right), & \text{if } k \text{ is even}. \end{cases}$$

Note that $\cos\left(\dfrac{(n-1)k\pi}{4}\right) = 1$ if $n \equiv 1 \bmod 4$.

From (A), (B), and (C), we state our main theorem.

Theorem. *Suppose n is odd and square-free. Then*

$$2\sum_{j=0}^{\phi(n)/2-1} b_j + b_{\frac{\phi(n)}{2}} = \begin{cases} \sqrt{n^w}\varepsilon^{-h(n)}, & \text{if } n \equiv 1 \bmod 8, \\ \sqrt{n^w}\varepsilon^{3h(n)}, & \text{if } n \equiv 5 \bmod 8. \end{cases}$$

where b_j are determined by the following recurrence formula

$$b_0 = 1,$$

$$jb_j = -\sum_{i=0}^{j-1} Q_{j-i}b_i \ (Q_k \text{ is given in (C)}).$$

Here are some examples:

Example 1. $n = 17$. Then

$$L_{17}(x) = 1 + 9x^2 + 11x^4 - 5x^6 - 15x^8 - 5x^{10} + 11x^{12} + 9x^{14} + x^{16} - \sqrt{17}x(1 + 3x^2 + x^4 - 3x^6 - 3x^8 + x^{10} + 3x^{12} + x^{14}).$$

So $L_{17}(1) = \sqrt{17}(-4 + \sqrt{17}) = \sqrt{17}(4 + \sqrt{17})^{-1}$. On the other hand, $\varepsilon = 4 + \sqrt{17}$. Therefore we get $h(17) = 1$.

Example 2. $n = 21 = 3 \cdot 7$. Then

$$L_{21}(x) = 1 + 10x^2 + 13x^4 + 7x^6 + 13x^8 + 10x^{10} + x^{12} + \sqrt{21}x(1 + 3x^2 + 2x^4 + 2x^6 + 3x^8 + x^{10}).$$

So $L_{21}(1) = 55 + 12\sqrt{21} = \left(\dfrac{5 + \sqrt{21}}{2}\right)^3$. Since $\varepsilon = \dfrac{5 + \sqrt{21}}{2}$, $h(21) = 1$.

Example 3. $n = 65 = 5 \cdot 13$. Then

$$L_{65}(x) = 1 + 32x^2 + 138x^4 + 69x^6 - 290x^8 - 79x^{10} + \cdots - \sqrt{65}x(1 + 10x^2 + 19x^4 - 14x^6 - 38x^8 + 37x^{10} + \cdots).$$

Therefore $\tilde{G}_{65}(1) = 129 + 16\sqrt{65} = (8 + \sqrt{65})^2$ and $L_{65}(1) = 129 - 16\sqrt{65} = (8 + \sqrt{65})^{-2}$. Since $\varepsilon = 8 + \sqrt{65}$, $h(65) = 2$.

References

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