95. *Elliptic Factors of Selberg Zeta Functions*

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We show that the elliptic factors of Selberg zeta functions are expressed in terms of multiple gamma functions.

§1. Elliptic factors. Let $X = G/K$ be a rank one symmetric space of non compact type, where G is a connected semisimple Lie group with finite center, and K is a maximal compact subgroup of G. Put $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{k} = \mathrm{Lie}(K)$ and let $\mathfrak{g}_C$, $\mathfrak{k}_C$ be their complexifications. We assume that $\mathrm{rank}G = \mathrm{rank}K$ and fix a Cartan subgroup T of G which is contained in K. We choose a system of positive roots of $\Phi(\mathfrak{g}_C, \mathfrak{t}_C)$ and a singular imaginary root α_t in $\Phi^+(\mathfrak{g}_C, \mathfrak{t}_C)$. Let $G = KA_RN$ be an Iwasawa decomposition of G. From the assumption, A_R is a one dimensional real torus. We identify $\mathfrak{a}_R$ with $\boldsymbol{R}$ as in [11]. Let ρ_0 be the half of the sum of positive roots in $\Phi(\mathfrak{g}, \mathfrak{a}_R)$. Let M be the centralizer of A_R in K, and A_I be a Cartan subgroup of M. Let $\mathfrak{m}$, $\mathfrak{a}_R$, $\mathfrak{a}_I$ be the Lie algebras of M, A_R, A_I, and $\mathfrak{m}_C$, $\mathfrak{a}_{R,C}$, $\mathfrak{a}_{I,C}$ their complexifications respectively. Let Γ be a discrete subgroup of G such that $vol(G/\Gamma) < \infty$. We define the elliptic factor of the Selberg zeta function for (G, Γ) as a smooth function $Z_{ell}(s)$ on a half interval (a, ∞) of $\boldsymbol{R}$ which satisfies the identity

$$(*) \qquad \left(-\frac{1}{2(s-\rho_0)}\frac{d}{ds}\right)^m \log Z_{ell}(s) = \int_0^{+\infty} I_{ell}(h_t)e^{-s(s-2\rho_0)t}t^{m-1}dt$$

for a positive integer m, where I_{ell} denotes the elliptic term of the Selberg trace formula for (G, Γ), and h_t is the spherical fundamental solution of the heat equation $\left(\Delta + \frac{\partial}{\partial t}\right)u = 0$ on X. By this definition $Z_{ell}(s)$ is determined up to a factor $exp(P(s-\rho_0))$, where $P(s)$ is an even polynomial. We calculate the right hand side of $(*)$ using the Fourier inversion formula of elliptic orbital integrals [11], and determine $Z_{ell}(s)$ as a finite product of multiple gamma functions.

§2. Results. Let $\mathscr{E}_\Gamma$ be the set of elliptic conjugacy classes of Γ, consisting of all conjugacy classes of finite orders. For $\gamma \in \mathscr{E}_\Gamma$, we denote by n_γ its order. We choose an element t_γ of T which is conjugate to γ in G; t_γ is unique up to the action of the Weyl group $W = W(G, T)$. Let G_γ be the centralizer of t_γ in G and $\mathfrak{g}_\gamma$ be its Lie algebra. We write Φ_γ^+, Φ_I^+ the sets of all positive roots in $\Phi(\mathfrak{g}_{\gamma,C}, \mathfrak{t}_C)$, $\Phi_I = \Phi(\mathfrak{m}_C, \mathfrak{a}_{I,C})$ and r_γ, r_I their cardinalities respectively. For each element $w \in W$, put

$$P_{\gamma,w}(\nu) = \prod_{\beta\in\Phi_\gamma^+} (w(-\rho_I + \nu\alpha_t), \beta) \quad (\nu \in \boldsymbol{R}),$$

where ρ_I is the half of the sum of roots in Φ_I^+. For an integral $\lambda \in \sqrt{-1}\mathfrak{t}^*$, denote ξ_λ the corresponding unitary character of T. We define complex numbers $\theta_r^{(j)}(\gamma)\,(1 \leqslant j \leqslant r_\gamma)$ as follows:

$$\theta_r^{(j)}(\gamma) = d(\gamma)^{-1}(-1)^{q_\gamma} \sum_{w \in W/W_\gamma} (-1)^{l(w)} P_{\gamma,w}^{(j)}\left(\frac{r}{2}\right)\xi_{w\left(-\rho_I+\frac{r}{2}\alpha_t\right)}(t_\gamma),$$

where

$$d(\gamma) = |W_\gamma^{\boldsymbol{C}}|\,[G_\gamma : G_\gamma^0]\left(\prod_{\beta\in\Phi_\gamma^+}(\rho_\gamma, \beta)\right)\xi_\rho(t_\gamma) \prod_{\beta\in\Phi^+-\Phi_\gamma^+}(1-\xi_{-\beta}(t_\gamma)),$$

r is an integer such that $-\rho_I + \frac{r}{2}\alpha_t$ is a $\Phi(\mathfrak{g}_{\boldsymbol{C}}, \mathfrak{t}_{\boldsymbol{C}})$-integral element, $P_{\gamma,w}^{(j)}$ means j-th derivative of $P_{\gamma,w}$, $W_\gamma = W(G_\gamma^0, T)$, $W_\gamma^{\boldsymbol{C}} = W(\mathfrak{g}_{\gamma,\boldsymbol{C}}, \mathfrak{t}_{\boldsymbol{C}})$, $q_\gamma = \frac{1}{2} dim(G_\gamma/K_\gamma)$, $q = \frac{1}{2} dim(G/K)$, and $\rho_\gamma = \frac{1}{2}\sum_{\beta\in\Phi_\gamma^+}\beta$. We take the Euler Poincaré measure on G_γ as in [3]. We now state our main theorem.

Theorem. (1) *There exists a smooth function $Z_{ell}(s)$ which satisfies (*), and it is expressed in $Re(s) > \rho_0$ as follows:*

$$Z_{ell}(s) = \prod_{\gamma\in\mathscr{E}_\Gamma} Z_\gamma(s)^{vol(\Gamma_\gamma\backslash G_\gamma)},$$

where

$$Z_\gamma(s) = \prod_{\substack{0\leqslant r<2n_\gamma \\ r\equiv\varepsilon\,(\mathrm{mod}\,2)}} \left(\prod_{j=0}^{r_\gamma} G_{j+1}\left(\frac{s-\rho_0+r}{2n_\gamma}\right)^{c_r^{(j)}(\gamma)}\right),$$

$$c_r^{(j)}(\gamma) = \frac{(-1)^{r_I} n^j}{j!}\left(\theta_{-r}^{(j)}(\gamma)(-1)^{j+1} + \theta_r^{(j)}(\gamma)\right),$$

$$\varepsilon = \begin{cases} 0 & (G = SU(2n, 1)). \\ 1 & (otherwise) \end{cases}$$

$G_j(z)$ is a multiple gamma function of Barnes, and defined by means of Weierstrass product as follows ([1], [8], [9]).

$$G_{j+1}(s)^{-1} = exp\left(\frac{(-1)^j\gamma}{j+1}s^{j+1} + \frac{(-1)^j}{j}s^j\right)\prod_{n\geqslant 1} P_{j+1}\left(\frac{-s}{n}\right)^{n^j},$$

where $P_j(x) = (1-x)exp\left(x + \frac{x^2}{2} + \cdots + \frac{x^j}{j}\right)$, and γ is the Euler constant.

(2) *When Γ is cocompact, the above $Z_{ell}(s)$ has a meromorphic continuation to the whole complex plane and has possible simple poles or zeros at $s = \rho_0 + l$ for $l \in \boldsymbol{Z}$ with $l \equiv \varepsilon \bmod 2$. The order of zero at $s = \rho_0 + l$ is given by*

$$(-1)^{r_I}\sum_{\gamma\in\mathscr{E}_\Gamma} vol(\Gamma_\gamma\backslash G_\gamma)\left(\theta_{-l}^{(0)}(\gamma) - \theta_l^{(0)}(\gamma)\right).$$

Remarks. (1) An explicit form of $Z_\gamma(s)$ for central γ's have been obtained by Kurokawa by different method ([6], [7]).

(2) For cocompact Γ, the number

$$\sum_{\gamma\in\mathscr{E}_\Gamma} vol(\Gamma_\gamma\backslash G_\gamma)\,\theta_{-l}^{(0)}(\gamma),$$

appearing in the above formulae is, up to a sign, identified with an alternating sum of dimensions of certain L^2-cohomology spaces in [3].

(3) We define the completed Selberg zeta function $\hat{Z}(s)$ as follows:

$$\hat{Z}(s) = Z_{id}(s)Z_{ell}(s)Z_{par}(s)Z_\Gamma(s),$$

where the first three factors of the right hand side are regarded as gamma factors of the zeta function. According to the recent results of Jorgenson and Lang [4], one can obtain determinant expression of $\hat{Z}(s)$ under the assumption that its gamma factors are themselves expressed as regularized products. This assumption is satisfied for the identity factor and the elliptic factor by the results of Kurokawa [7] and ours. By using the results of Reznikov [10], the parabolic factor also has determinant expression when G is $SO(2n, 1)$ $(n \geqslant 1)$ or $SU(2n, 1)$ $(n \geqslant 1)$ and Γ is its congruence subgroup. The regularized products expression of $\hat{Z}(s)$ are known in some special cases [5].

§3. Proof of the theorem. By definition, we have

$$I_{ell}(h_t) = \sum_{\gamma \in \mathscr{E}_\Gamma} vol(\Gamma_\gamma \backslash G_\gamma) \mathcal{O}_\gamma(h_t) \quad (t > 0),$$

where $\mathcal{O}_\gamma(h_t) = \int_{G_\gamma \backslash G} h_t(x^{-1} t_\gamma x) \dfrac{dx}{dx_\gamma}$.

From the Fourier inversion formula of [11], $\mathcal{O}_\gamma(h_t)$ can be written as a finite linear combination of integrals of following type:

$$h_\theta^\varepsilon(P\,;t) = \int_{-\infty}^{+\infty} 2\hat{h}_t(\nu) \frac{e^{\nu\theta} P\left(\frac{\sqrt{-1}\,\nu}{2}\right) - (-1)^\varepsilon e^{-\nu\theta} P\left(-\frac{\sqrt{-1}\,\nu}{2}\right)}{e^{\pi\nu/2} - (-1)^\varepsilon e^{-\pi\nu/2}} d\nu,$$

where P is a polynomial with coefficients in $\boldsymbol{R}$, $\varepsilon \in \{0, 1\}$, and θ is a real number in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Hence, the calculation of $Z_{ell}(s)$ is reduced to that of the integrals

$$\Psi_\gamma(s, m) = \int_0^\infty e^{-s(s-2\rho_0)t} \mathcal{O}_\gamma(h_t) t^{m-1} dt,$$

and

$$I_\theta^\varepsilon(P\,; s, m) = \int_0^\infty e^{-t(s-\rho_0)^2} h_\theta^\varepsilon(P\,;t) t^{m-1} dt.$$

This is given as follows. Let $\theta = \dfrac{l\pi}{n}$ $(l, n \in \boldsymbol{Z}, -\pi \leqslant 2\theta \leqslant \pi)$, and $\varepsilon \in \{0, 1\}$. We first obtain

$$I_\theta^\varepsilon(P\,; s, m) = \left(-\frac{1}{2(s-\rho_0)} \frac{d}{ds}\right)^{m-1} I_\theta^\varepsilon(P\,; s)$$

$$2(s-\rho_0) I_\theta^\varepsilon(P\,; s) = \frac{\sqrt{-1}}{n} \sum_{\substack{0 \leqslant r < 4n \\ r \equiv \varepsilon \,(\mathrm{mod}\ 2)}} \psi\left(\frac{s-\rho_0+r}{4n}\right)$$
$$\times \left\{P\left(\frac{s-\rho_0}{2}\right) e^{-\frac{\pi-2\theta}{2} r\sqrt{-1}} - P\left(\frac{\rho_0 - s}{2}\right) e^{\frac{\pi-2\theta}{2} r\sqrt{-1}}\right\},$$

where $\psi(s)$ is the logarithmic derivative of the gamma function, and $m > deg(P)$. Let $\psi_{j+1}(s) = \dfrac{d}{ds} \log G_{j+1}(s)$ $(s \gg 0)$, then $Q_j(s) = \psi_{j+1}(s) - s_j \psi(s)$ is a polynomial. So it holds that

$$P\left(\frac{\sigma}{2}\right) \psi\left(\frac{\sigma+i}{2n}\right) = \sum_{j=0}^{deg(P)} \frac{(-n)^j}{j!} P^{(j)}\left(\frac{-i}{2}\right) \psi_{j+1}\left(\frac{\sigma+i}{2n}\right) + R(s)$$

for a polynomial $R(s)$. Thus we have

$$2(s-\rho_0)I_\theta^\varepsilon(P;s)=\frac{\sqrt{-1}}{n}\sum_{\substack{0\leqslant r<4n\\ r\equiv\varepsilon\,(\mathrm{mod}\,2)}}\sum_{j=0}^{deg(P)}\frac{(-n)^j}{j!}\psi_{j+1}\left(\frac{s-\rho_0+r}{4n}\right)$$
$$\times\left\{P(j)\left(\frac{-i}{2}\right)e^{-\frac{\pi-2\theta}{2}r\sqrt{-1}}-P^{(j)}\left(\frac{i}{2}\right)e^{\frac{\pi-2\theta}{2}r\sqrt{-1}}\right\}+R(s)$$

for a polynomial $R(s)$ and $m>deg(P)$. Now from the results of [11], we have

$$\Psi_\gamma(s,m)=\frac{\sqrt{-1}\,|W_\gamma|\,(-1)^{r_I}}{8\,|W(M,A_I)|\,d(\gamma)}\sum_{u\in W_I}(-1)^{l(u)}\sum_{w_i}(-1)^{l(w_i)}$$
$$\times\bar{\chi}_I^u(t_\gamma)(sgn\theta_i)^\varepsilon I^\varepsilon_{\theta_i+sgn\theta_i\frac{\pi}{2}}(P_{\gamma,w_i u};s,m),$$

where χ_I is the unitary character of A_I with differential ρ_I, $\{w_i\}$ is a complete set of representative of W_γ in W, and for each w_i, θ_i is the real number such that $\xi_{\alpha_t}(w_i^{-1}t_\gamma)=e^{-2\sqrt{-1}\,\theta_i}$ $(-\pi\leqslant 2\theta_i<\pi)$, and ε is given in the theorem. Consequently, we obtain $\Psi_\gamma(s,m)$ for $m>r_\gamma$ as follows.

$$\Psi_\gamma(s,m)=\left(-\frac{1}{2(s-\rho_0)}\frac{d}{ds}\right)^{m-1}\Psi_\gamma(s),$$

where

$$2(s-\rho_0)\Psi_\gamma(s)=\sum_{\substack{0\leq r<2n_\gamma\\ r\equiv\varepsilon\,(\mathrm{mod}\,2)}}\sum_{j=0}^{r_\gamma}c_\gamma^{(j)}\frac{1}{2n_\gamma}\psi_{j+1}\left(\frac{s-\rho_0+r}{2n_\gamma}\right).$$

This gives our expression of $Z_{ell}(s)$ via multiple gamma functions.
Since $\psi_{j+1}(s)$ has simple poles at $s=-n$ $(n\in\boldsymbol{Z},\ n\leqslant 0)$, with residue $-(-n)^j$,

$$\Psi(s)=\sum_{\gamma\in\mathscr{E}\Gamma}vol(\Gamma_\gamma\backslash G_\gamma)\Psi_\gamma(s)$$

has a possible simple pole at $s=\rho_0+l$ $(l\in\boldsymbol{Z},\ l\equiv\varepsilon\ (\mathrm{mod}\ 2))$ and

$$\operatorname*{Res}_{s=\rho_0+l}\Psi(s)=(-1)^{r_I}\sum_\gamma vol(\Gamma_\gamma\backslash G_\gamma)(\theta_{-l}^{(0)}(\gamma)-\theta_l^{(0)}(\gamma)).$$

When G/Γ is compact, this is an integer [3]. Thus we have a meromorphic $Z_{ell}(s)$ satisfying $\frac{d}{ds}\log Z_{ell}(s)=\Psi(s)$.

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