47. *On Multiplicative Semigroups of Von Neumann Regular Rings*

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§1. Introduction. We recall from [1] and [3] that

(1) a monoid M is *left [right] absolutely flat* if any left [right] M-set is flat, and it is *absolutely flat* if it is a both left and right absolutely flat, and

(2) a monoid M is *strongly left [right] reversible* if for any $x, y \in M$, there exists $z \in M$ such that $zx = x$ and $zy \in xM \cap yM$ [respectively, $xz = x$ and $yz \in Mx \cap My$], and it is *strongly reversible* if it is both left and right reversible.

Let S be a semigroup and S^1 the monoid obtained by adjoining a new identity 1 if S does not have an identity. Following [1], we say that a semigroup S is [*left, right*] *absolutely flat* if S^1 is a [left, right] absolutely flat monoid. Similarly, we say that a semigroup S is [*left, right*] *strongly reversible* if S^1 is a [left, right] strongly reversible.

In [3], Bulman-Fleming and McDowell proved that the multiplicative semigroup of any semi-simple Artinian ring is strongly reversible. This gives an impulse to us for proceeding to our result stated below.

Theorem. *Let R be any ring. Then the following are equivalent*:

(1) *R is a Von Neumann regular ring.*

(2) *The multiplicative semigroup of R is strongly reversible.*

(3) *The multiplicative semigroup of R is absolutely flat.*

Our proof of the theorem is simple and just a combination of a few basic facts concerning idempotents of regular rings.

A semigroup S is called a *semigroup amalgamation base* if for any family $\{T_i \mid i \in I\}$ of oversemigroups T_i of S, there exists a semigroup V in which each T_i is embedded with the property that intersection of every pair of T_i and T_j $(i \neq j)$ in V equals S.

It is well-known that absolutely flat semigroups are semigroup amalgamation bases (see [2]).

Here we have

Corollary. *The multiplicative semigroup of any Von Neumann regular ring is a semigroup amalgamation base.*

§2. A proof of theorem. In their paper [2], Bulman-Fleming and McDowell introduced V. Fleischer's characterization of absolutely flat monoids and pointed out that every strongly reversible monoid is absolutely flat. Thus the implication (2) $\Rightarrow$ (3) of Theorem is obtained. It follows from Kilp's theorem [5] (or [1, Proposition 2.5]) that every absolutely flat monoid is regular. Then the implication (3) $\Rightarrow$ (1) is proved. Therefore it suffices to prove the implication (1) $\Rightarrow$ (2).

Proof of the implication (1) $\Rightarrow$ (2). Let R be a regular ring. We shall show first that the monid R^1 is strongly left reversible. Let $x, y \in R^1$. If $x=1$, then $1x=x$, $1y \in xR^1 \cap yR^1$. If $x \in R$, $y=1$, then by [4, Theorem 1.1 (b)], there exists an idempotent $e \in R$ with $eR = xR$, such that $ex=x$, $ey \in xR^1 \cap yR^1$. Thus we can assume that $x, y \in R$. Then by [4, Theorem 1.1(b)], there exist idempotents $e, f \in R$ such that $xR^1 = eR$, $yR^1 = fR$. On the other hand, by [4, Lemma 2.2], the right ideal $xR^1 \cap yR^1$ is finitely generated and, again, by [4, Theorem 1.1(c)], there exists an idempotent $f_1 \in R$ such that $xR^1 \cap yR^1 = f_1R$. Put $f_2 = f - f_1f$. Then it is easily seen that f_2 is an idempotent and $eR \cap f_2R = 0$. By [4, Theorem 1.1(c)], there exists an idempotent $h \in R$ such that $xR + yR = hR$. Since $eR \oplus f_2R = xR^1 + yR^1$, there exist $s, t \in R$ such that $es + f_2t = h$. Then $e = he = ese + f_2te$. This implies that $e = ese$, since $eR \cap f_2R = 0$. Hence es is an idempotent and $esR = xR^1$. Similarly we get $f_2 = esf_2 + f_2tf_2$, which implies $esf_2 = 0$. Put $z = es$. Then $zx = (es)(ex) = ex = x$ and $zy = (es)(fy) = (es)(f_1f + f_2)y = es(f_1f)y = es(ef_1)fy = f_1(fy) \in xR^1 \cap yR^1$, as required. Hence R is strongly left reversible. By symmetry, it is shown that R is strongly right reversible, and strongly reversible. Therefore the implication (1) $\Rightarrow$ (2) is proved.

Remark. In the latter part of the proof of the implication (1) $\Rightarrow$ (2), it is shown that for idempotents e, f_2 of a regular ring R with $eR \cap f_2R = 0$, there exists an idempotent z such that $ze = e$, $zf_2 = 0$. This was already suggested by Utsumi [6, Section 3, p. 159].

References

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