97. *Selberg Trace Formula for Odd Weight.* I

By Shigeki AKIYAMA
Department of Mathematics, Kobe University

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§0. **Introduction.** Let Γ be a fuchsian group of the first kind which does not contain the element $-I=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$. The first aim of this note is to rewrite the Selberg trace formula for odd weight and the group Γ, in a form which makes clear the difference between the contributions of regular cusps and of irregular cusps. Such a difference is already known in the case of the dimension formula for weight ≥ 3 and we are interested to know how it appears in general case. Our second aim is to improve the dimension formula for weight one. At the symposium "Automorphic Forms and Related Topics" at Research Institute of Mathematical Science, Kyoto University on 1987, Tanigawa, Hiramatsu and the author made a report on the trace formula for Hecke operators in the case of weight one. In the special case of the report, we gave a dimension formula of the space of cusp forms of weight one, using the residue of the Selberg type zeta function ([5]). But the formula is unsatisfactory because the "zeta" has no functional equation. In this note, we give a dimension formula of weight one, using more natural "zeta" function which has a functional equation.

§1. **Notation.** Let $\boldsymbol{H}$ be the complex upper half plane and $T=\boldsymbol{R}/(2\pi)$. We put $\tilde{\boldsymbol{H}}=\boldsymbol{H}\times T$, $G=SL(2,\boldsymbol{R})$ and $\tilde{G}=G\times T$. Then $\tilde{G}$ acts transitively on $\tilde{\boldsymbol{H}}$ in the following way:

$$(g,\alpha)\cdot(z,\phi)=(g\cdot z,\phi+\arg j(g,z)-\alpha)\qquad (g,\alpha)\in\tilde{G},\quad (z,\phi)\in\tilde{\boldsymbol{H}},$$

where $g=\begin{bmatrix}a & b\\ c & d\end{bmatrix}\in G$, $g\cdot z=\dfrac{az+b}{cz+d}\in\boldsymbol{H}$ and $j(g,z)=cz+d$. With the involution $\xi(z,\phi)=(-\bar{z},-\phi)$, the triple $(G,\boldsymbol{H},\xi)$ is the weakly symmetric Riemannian space. The ring of $\tilde{G}$ invariant differential operators on this space is generated by Δ and $\partial/\partial\phi$ where

$$\Delta=y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+y\frac{\partial}{\partial x}\frac{\partial}{\partial\phi}.$$

Let Γ be the discrete subgroup of G not containing $-I$. We identify G with $G\times\{0\}$, and Γ with $\Gamma\times\{0\}$. Take a unitary representation χ of Γ of degree ν $(<\infty)$. Let $\kappa_1,\kappa_2,\cdots,\kappa_\omega$ be the complete representatives of Γ-inequivalent cusps of $\Gamma\backslash\boldsymbol{H}$. Γ_i denotes the stabilizer of κ_i, and $\Gamma_i^0=\Gamma_i\cap\ker\chi$. We will consider the cusps form of $\Gamma\backslash\boldsymbol{H}$, so take χ under the condition $[\Gamma_i\Gamma_i^0]<\infty$ for $i=1,2,\cdots,\omega$. Take $\sigma_i\in G$ such that $\sigma_i\infty=\kappa_i$, satisfying the following condition:

If κ_i is regular then $\Gamma_\infty=\sigma_i^{-1}\Gamma_i\sigma_i$ is generated by $\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$;

If κ_i is irregular then $\Gamma_\infty=\sigma_i^{-1}\Gamma_i\sigma_i$ is generated by $\begin{bmatrix}-1 & -1\\ 0 & -1\end{bmatrix}$.

We consider C^ν valued, square integrable functions on $\tilde{H}$ satisfying $f(\gamma\cdot(z,\phi))=\chi(\gamma)f((z,\phi))$ for any $\gamma\in\Gamma$, and denote the space which consists of these functions as $L^2_\chi(\Gamma,\tilde{H})$. Selberg eigenspace is a subspace of $L^2_\chi(\Gamma,\tilde{H})$ defined by two additional conditions:

$$\begin{cases} 1) & \dfrac{\partial}{\partial\phi}f=-\sqrt{-1}\,mf, \\ 2) & \Delta f=\lambda f. \end{cases}$$

$\mathcal{L}_\chi(m,\lambda)$ denotes this space. We assume the eigen values λ are numbered in the following way

$$\frac{|m|}{2}\left(\frac{|m|}{2}-1\right)\geq\lambda_1\geq\lambda_2\geq\lambda_3\geq\cdots.$$

For the convenience, we set $\lambda_n=-(r_n^2+1/4)=s_n(s_n-1)$ with $s_n=1/2+\sqrt{-1}\,r_n$. To describe the continuous spectrum, we define the Eisenstein series which attaches to κ_i $(i=1,\cdots,\omega)$:

$$\begin{aligned} E_i(z,\phi;s)&=\sum_{\sigma\in\Gamma_i\backslash\Gamma}\operatorname{Im}(\sigma_i^{-1}\sigma z)^s\exp(-m\sqrt{-1}\,\sigma_i^{-1}\sigma\phi)\chi^{-1}(\sigma)P_i \\ &=\sum_\sigma\frac{y^s}{|cz+d|^{2s}}\exp(-m\sqrt{-1}\,(\phi+\arg(cz+d)))\chi^{-1}(\sigma_i\sigma)P_i. \end{aligned}$$

In the last summation, σ is taken over all representatives of $\Gamma_\infty\backslash\sigma_i^{-1}\Gamma$, and $\sigma=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$. P_i is defined by

$$P_i=\begin{cases}\dfrac{1}{r_i}\displaystyle\sum_{g\in\Gamma_i/\Gamma_i^0}\chi(g) & (\text{if }\kappa_i\text{ is regular}); \\ \dfrac{1}{2r_i}\displaystyle\sum_{i=1}^{2r_i}(-1)^i\chi(\eta^i) & (\text{if }\kappa_i\text{ is irregular}),\end{cases}$$

where $r_i=[\Gamma_i:\Gamma_i^0]$ and $\eta\in\Gamma_i$ is chosen so that $\eta \bmod (\Gamma_i^0)^2$ should be a generator of $\Gamma_i/(\Gamma_i^0)^2$. These Eisenstein series are meromorphically continued to the whole s-plane and satisfies a certain functional equation (cf. [2], [6]). We denote by $\Phi_m(s)$ the constant term matrix of these Eisenstein series.

§2. Selberg trace formula for odd weight. First of all we rewrite the Selberg trace formula in our case. The calculation was done by Hejhal in [1], but we do this by our formulation of the Eisenstein series, using the Euler-Maclaurin summation formula.

Theorem 1 (Selberg trace formula for odd weight). *Let N be a non negative integer and $m=2N+1$. We assume that $h(r)$ is an analytic function in the region $|\operatorname{Im}(r)|\leq\max(N,1/2)+\delta$ satisfying following two conditions:*

1) $h(r)=h(-r)$;
2) *There exists a sufficiently large number M such as*
$$h(r)\leq M|1+\operatorname{Re}(r)|^{-2-\delta},$$

where δ is some positive real number. Put $g(u)=1/2\pi\int_{-\infty}^{\infty}h(r)e^{-\sqrt{-1}\pi ru}dr$, then the following formula holds:

$$
\begin{aligned}
\sum_{n=1}^{\infty} h(r_n) = & \frac{\nu \operatorname{vol}(\Gamma \backslash \boldsymbol{H})}{4\pi} \left[\int_{-\infty}^{\infty} r h(r) \coth(\pi r) dr + 2 \sum_{k=0}^{N} k \ h(\sqrt{-1}k) \right] \\
& + \sum_{\{T\} \in \text{hyperbolic}} \frac{\operatorname{Tr}(\chi(T)) \operatorname{sgn}(T) \ln N\{T_0\}}{N\{T\}^{1/2} - N\{T\}^{-1/2}} g(\ln N\{T\}) \\
& + \sum_{\{R\} \in \text{elliptic}} \frac{\operatorname{Tr}(\chi(R))}{4{}^{\#}\Gamma(R) \sin\theta} \left[\int_{-\infty}^{\infty} h(r) \frac{\sinh(\pi - 2\theta) r}{\sinh \pi r} dr \right. \\
& \left. - \sqrt{-1} h(0) + 2\sqrt{-1} \sum_{k=0}^{N} e^{2\sqrt{-1}k\theta} h(\sqrt{-1}k) \right] \\
& - g(0) \sum_{\substack{\alpha_{ij} \neq 0 \\ \text{regular}}} \ln |1 - e^{2\pi i \alpha_{ij}}| + \sum_{\substack{\alpha_{ij} \neq 0 \\ \text{regular}}} \left(\frac{1}{2} - \alpha_{ij} \right) \left(\sum_{k=0}^{N} h(\sqrt{-1}k) - \frac{h(0)}{2} \right) \\
& - g(0) \sum_{\substack{\alpha_{ij} \neq -1/2 \\ \text{irregular}}} \ln |1 + e^{2\pi i \alpha_{ij}}| - \sum_{\substack{\alpha_{ij} \neq -1/2 \\ \text{irregular}}} \alpha_{ij} \left(\sum_{k=0}^{N} h(\sqrt{-1}k) - \frac{h(0)}{2} \right) \\
& + \frac{l}{4} h(0) - \frac{h(0)}{4} \operatorname{Tr} \left(\Phi_m \left(\frac{1}{2} \right) \right) \\
& + \frac{1}{4\pi} \int_{-\infty}^{\infty} h(r) \operatorname{Tr} \left(\Phi_m' \left(\frac{1}{2} + \sqrt{-1} r \right) \Phi_m \left(\frac{1}{2} - \sqrt{-1} r \right) \right) dr - l g(0) \ln 2 \\
& - \frac{l}{2\pi} \int_{-\infty}^{\infty} h(r) \psi(1 + \sqrt{-1} r) dr + l \int_0^{\infty} g(u) \frac{1 - \cosh(mu/2)}{2 \sinh(u/2)} du.
\end{aligned}
$$

Here the notation is as follows. We denote by "$\sim$" the conjugation in $SL(2, \boldsymbol{R})$ and by $\{\ \}$ its conjugate class. Take λ such that $T \sim \begin{bmatrix} \lambda & \\ & \lambda^{-1} \end{bmatrix}$, where $|\lambda| > 1$; $\Gamma(T)$ denotes the centralizer of T in Γ and ${}^{\#}\Gamma(T)$ is the order of this group. T_0 is a generator of $\Gamma(T)$, $N\{T\} = \lambda^2$ and $\operatorname{sgn} T = \operatorname{sgn} \lambda$. $R \sim \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$; T_i is a generator of Γ_i, the stabilizer group of cusp κ_i.

$$
\chi(T_i) \sim \begin{bmatrix} \exp(2\pi\sqrt{-1}\alpha_{i1}) & & \\ & \ddots & \\ & & \exp(2\pi\sqrt{-1}\alpha_{i\nu}) \end{bmatrix}.
$$

We determine α_{ij} so that

$$
\begin{cases} \alpha_{ij} \in [0, 1) & \text{if } \kappa_i \text{ is regular} \\ \alpha_{ij} \in [-1/2, 1/2) & \text{if } \kappa_i \text{ is irregular.} \end{cases}
$$

$l = A + B$ where A denotes the number of pairs (i, j) such that $\alpha_{ij} = 0$, where i moves in the range that κ_i is regular and $j = 1, \cdots, \nu$. And B denotes the number of pairs (i, j) such that $\alpha_{ij} = -1/2$, where i moves in the range that κ_i is irregular and $j = 1, \cdots, \nu$. $\psi(s) = \Gamma'/\Gamma(s)$ is the digamma function.

§ 3. Selberg zeta function for odd weight. Put

$$
Z_\Gamma^*(s, \chi) = \prod_{\alpha} \prod_{n=0}^{\infty} \det(E_\nu - \operatorname{sgn}(P_\alpha) \chi(P_\alpha) N\{P_\alpha\}^{-s-n})
$$

where the first product $\prod$ is taken over all primitive hyperbolic conjugate classes $\{P_\alpha\}$ of Γ, and E_ν is the $\nu \times \nu$ unit matrix. Now we can write down the functional equation. We put

$$
\begin{aligned}
\zeta_\Gamma^*(s, \chi) &= \frac{d}{ds} \log Z_\Gamma^*(s, \chi) \\
&= \sum_{\alpha} \sum_{k=1}^{\infty} \frac{\operatorname{Tr}(\chi(P_\alpha^k)) \operatorname{sgn}(P_\alpha^k) \ln N\{P_\alpha\}}{N\{P_\alpha\}^{k/2} - N\{P_\alpha\}^{-k/2}} N\{P_\alpha\}^{-(s-1/2)k}.
\end{aligned}
$$

Theorem 2 (Functional equation). *We have*

$$\zeta_\Gamma^*(s,\chi)+\zeta_\Gamma^*(1-s,\chi)=-\nu\,\mathrm{vol}\,(\Gamma\backslash\boldsymbol{H})(s-1/2)\cot(\pi(s-1/2))$$
$$-\pi\sum_{\{R\}}\frac{\mathrm{Tr}\,(\chi(R))}{{}^{\#}\Gamma(R)\sin\theta}\frac{\sin((\pi-2\theta)(s-1/2))}{\sin(\pi(s-1/2))}$$
$$+2\sum_{\substack{\alpha_{ij}\neq 0\\ \text{regular}}}\ln|1-e^{2\pi\sqrt{-1}\alpha_{ij}}|+2\sum_{\substack{\alpha_{ij}\neq -1/2\\ \text{irregular}}}\ln|1+e^{2\pi\sqrt{-1}\alpha_{ij}}|$$
$$-\frac{\varphi'}{\varphi}(s)-\frac{l}{2}(\xi_m(s)+\xi_m(1-s))+2l\ln 2$$

where $\xi_m(s)=\psi(s+m/2)+\psi(s-m/2)-2\psi(s)-2\psi(s+1/2)$, $\varphi(s)=\det\Phi_m(s)$.

(to be continued.)

References

[1] Hejhal, D. A.: The Selberg trace formula for $PSL(2,\boldsymbol{R})$. I, II. Lecture Notes in Math., vol. 548, 1001, Springer, Berlin (1976, 1983).

[2] H. Ishikawa and Y. Tanigawa: The dimension formula of the space of cusp forms of weight one for $\Gamma_0(p)$. Proc. Japan Acad., **63**A, 31–34 (1987).

[3] Selberg, A.: Harmonic analysis and discontinuous groups in weakly symmetric Riemannian space with application to Dirichlet series. J. Indian Math. Soc., **20**, 47–87 (1956).

[4] ——: Discontinuous groups and harmonic analysis. Proc. International Congr. Math., pp. 177–189 (1962).

[5] S. Akiyama, T. Hiramatsu, and Y. Tanigawa: On the traces of Hecke operators on the space of cusp forms of weight one (preprint).

[6] T. Hiramatsu and S. Akiyama: On some dimension formula for automorphic forms of weight one. III. Nagoya Math. J., **111** (1988).

[7] T. Hiramatsu: On some dimension formula for automorphic forms of weight one. I, II. ibid., **85**, 213–221 (1982); ibid., **105**, 169–186 (1987).

[8] T. Kubota: Elementary Theory of Eisenstein Series. Kodansha and Halsted, Tokyo-New York (1973).