109. On Periodic Solutions for the Periodic Quasilinear Ordinary Differential System Containing a Parameter

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1. Introduction. In this paper we deal with the dependence on a parameter λ of T-periodic solutions for the T-periodic quasilinear ordinary differential system:

(1) $$x'=A(t,x,\lambda)x+\lambda F(t,x,\lambda)+f(t).$$

Here A is a real $n\times n$ matrix and F is an $\boldsymbol{R}^n$-valued function. We assume that A and F are defined on $\boldsymbol{R}\times\boldsymbol{R}^n\times[-\lambda_0, +\lambda_0]$, continuous in (t,x,λ) and T-periodic in t, where $\lambda_0>0$. We assume that f is an $\boldsymbol{R}^n$-valued function continuous on $\boldsymbol{R}$ and T-periodic.

We consider the associated T-periodic linear system:

(2) $$x'=B(t)x+f(t),$$

where B is a real $n\times n$ matrix continuous on $\boldsymbol{R}$ and T-periodic.

Hypothesis 1. *For every f continuous on $\boldsymbol{R}$ and T-periodic, there exists one and only one T-periodic solution for* (2).

The qualitative studies of solutions for the periodic quasilinear differential system have been made under Hypothesis 1 (see [1], [2]). When λ is sufficiently small, Cronin [1] has discussed the existence of T-periodic solutions for

(3) $$x'=B(t)x+\lambda F(t,x,\lambda)+f(t)$$

by applying the implicit function theorem. When the Lipschitz conditions are satisfied, Hale [2] has dealt with the continuous dependence on λ of the T-periodic solution for (3) under some additional assumptions.

Theorem 1 in the present paper is the existence theorem of periodic solutions for periodic linear systems which are close to (2) in some sense. Theorem 2 is a strict extension of the standard result (see [1]). Moreover we give an extent that shows how A in (1) is close to B in (2) as well as an extent that shows how small λ is. In Theorem 3 we obtain sufficient conditions for some dependence on λ of periodic solutions for (1). Explicit conditions in Theorem 4 ensure the continuous dependence on λ of the periodic solution for (1).

2. Preliminaries. The symbol $\|\cdot\|$ will denote a norm in $\boldsymbol{R}^n$ and the corresponding norm for $n\times n$ matrices. Let C_T be the space of $\boldsymbol{R}^n$-valued functions continuous on $\boldsymbol{R}$ and T-periodic with the supremum norm. Let $C[0,T]$ be the space of $\boldsymbol{R}^n$-valued functions continuous on $[0,T]$ with the supremum norm $\|\cdot\|_\infty$.

We define a bounded linear operator $\mathcal{L}: C[0,T]\to \boldsymbol{R}^n$ by $\mathcal{L}(x(\cdot))=x(0)-x(T)$ with the norm

$$\|\mathcal{L}\|=\sup\{\|\mathcal{L}(x(\cdot))\|\,;\,\|x\|_\infty=1\}.$$

Let X_B be the fundamental matrix of solutions for the homogeneous system corresponding to (2) such that $X_B(0)=I$, where I is the identity matrix. Put $U_B=I-X_B(T)$, we have $\mathcal{L}(X_B(\cdot)x_0)=U_Bx_0$ for $x_0\in \boldsymbol{R}^n$.

The following lemmas are well known.

Lemma 1. *Hypothesis* 1 *is equivalent to* $\det U_B\neq 0$ (*see* [2]).

Lemma 2. *If* $\det U_B\neq 0$, *then we can choose a positive constant* ρ $(0<\rho<1)$ *such that*

$$\|U_B^{-1}\|\leqq 1/\rho. \tag{4}$$

Suppose that Hypothesis 1 holds, and fix a positive number ρ satisfying (4). We put $r_0=MK(1+2K/\rho)$, where

$$M=\int_0^T\|f(s)\|\,ds \quad \text{and} \quad K=\exp\left(\int_0^T\|B(s)\|\,ds\right).$$

Let $S_r=\{x\in \boldsymbol{R}^n\,;\,\|x\|\leqq r\}$ and let $C_{T,r}=\{y\in C_T\,;\,\|y\|_\infty\leqq r\}$, where $r>r_0$. Now we assume that three positive numbers δ, Δ and λ_1 $(\lambda_1\leqq\lambda_0)$ satisfy the conditions (5)–(6) below.

$$K^2\delta\exp(K\delta)\leqq\rho/\{2\|U_B^{-1}\|\}. \tag{5}$$

$$\{\lambda_1\Delta+M\}K\exp(\delta)[1+2K\exp(\delta)/\{\rho(1-\rho)\}]\leqq r. \tag{6}$$

We assume that A, F satisfy the conditions (7)–(8), respectively.

$$\int_0^T\|A(s,x,\lambda)-B(s)\|\,ds\leqq\delta \qquad \text{for } x\in S_r,\ \lambda\in\Lambda_1. \tag{7}$$

$$\int_0^T\|F(s,x,\lambda)\|\,ds\leqq\Delta \qquad \text{for } x\in S_r,\ \lambda\in\Lambda_1. \tag{8}$$

Here $\Lambda_1=[-\lambda_1, +\lambda_1]$.

3. Theorems. First, we consider the periodic linear non-homogeneous system:

$$x'=A(t,y(t),\lambda)x+\lambda F(t,y(t),\lambda)+f(t) \qquad \text{for } y\in C_{T,r} \tag{9}$$

together with a boundary condition

$$\mathcal{L}(x)=0 \qquad \text{for } x\in C[0,T], \tag{10}$$

where $\lambda\in\Lambda_1$. Put $U_y=I-X_y(T)$, where X_y is the fundamental matrix of solutions for the linear homogeneous system corresponding to (9) such that $X_y(0)=I$, we have $\mathcal{L}(X_y(\cdot)x_0)=U_yx_0$ for $x_0\in \boldsymbol{R}^n$.

Theorem 1. *Suppose that Hypothesis* 1 *holds and that the conditions* (5)–(8) *are satisfied. Then, for any* $y\in C_{T,r}$ *and any* $\lambda\in\Lambda_1$, *there exists the inverse of* U_y *such that*

$$\|U_y^{-1}\|\leqq 1/\{\rho(1-\rho)\} \tag{11}$$

and there exists one and only one solution $x_y\in C_{T,r}$ *for* ((9), (10)) *such that*

$$x_y(t)=-U_y^{-1}[\mathcal{L}(p_y(\cdot))]+\int_0^t A(s,y(s),\lambda)x_y(s)ds$$
$$+\lambda\int_0^t F(s,y(s),\lambda)ds+\int_0^t f(s)ds \qquad \text{for } t\in \boldsymbol{R},$$

where $p_y(t)=X_y(t)\int_0^t X_y^{-1}(s)\{\lambda F(s,y(s),\lambda)+f(s)\}ds$ *for* $t\in \boldsymbol{R}$.

This theorem is proved in the same manner as given in the proof of Theorem in [3].

From the above, we obtain the existence theorem of periodic solutions for (1).

Theorem 2. *Suppose that Hypothesis* 1 *holds. If the conditions* (5)–(8) *are satisfied, then for any* $\lambda \in \Lambda_1$ *there exists at least one T-periodic solution for* (1).

Sketch of the proof of Theorem 2. Choose $\lambda \in \Lambda_1$. From Theorem 1 we can define $\mathcal{G}: C_{T,r} \to C_{T,r}$ by $[\mathcal{G}(y)](t) = x_y(t)$ for $t \in \boldsymbol{R}$, where x_y is the T-periodic solution for (9) in $C_{T,r}$. It can be easily seen that $\mathcal{G}$ is a compact continuous operator. By Schauder's fixed point theorem, $\mathcal{G}$ has at least one fixed point in $C_{T,r}$. Thus for $\lambda \in \Lambda_1$ there exists at least one T-periodic solution for (1). Q.E.D.

Now we assume that the following hypothesis holds.

Hypothesis 2. *There exists a continuous and strictly increasing function* $\mu : [0, \lambda_2] \to \boldsymbol{R}^+$ $(0 < \lambda_2 \leqq \lambda_1)$ *such that* $\mu(0) = 0$ *and that*

$$\|A(t, x, \lambda) - B(t)\| \leqq \mu(|\lambda|) \qquad \text{for } (t, x, \lambda) \in [0, T] \times S_r \times \Lambda_2,$$

where $\Lambda_2 = [-\lambda_2, +\lambda_2]$ *and* $\boldsymbol{R}^+ = [0, +\infty)$.

Then we have the following theorem.

Theorem 3. *If, under the assumption in Theorem* 2, *Hypothesis* 2 *holds, then for any* $\varepsilon > 0$ *there exists an* $\eta(\varepsilon) > 0$ *such that for all* λ, $|\lambda| \leqq \eta(\varepsilon)$, *there exists at least one T-periodic solution* $x(\cdot\,; \varepsilon, \lambda)$ *for* (1) *satisfying*

$$(12) \qquad \|x(t; \varepsilon, \lambda) - \pi(t)\| \leqq \varepsilon \qquad \text{for } t \in \boldsymbol{R},$$

where π *is the T-periodic solution for* (2).

Sketch of the proof of Theorem 3. Choose ε such that $0 < \varepsilon < r - r_0$. Let $\eta = \eta(\varepsilon)$ satisfy the following inequality:

$$\{r_0 T \mu(\eta) + \eta \Delta\} K \exp(\delta)[1 + 2K \exp(\delta)/\{\rho(1-\rho)\}] \leqq \varepsilon$$

and let $C_{T,\varepsilon} = \{y \in C_T \,;\, \|y\|_\infty \leqq \varepsilon\}$. Choose λ such that $|\lambda| \leqq \eta(\varepsilon)$.

We consider the following linear non-homogeneous system:

$$(13) \qquad z' = A_1(t, y(t), \lambda)z + \lambda F_1(t, y(t), \lambda) + f_1(t, y(t), \lambda) \qquad \text{for } y \in C_{T,\varepsilon}$$

together with a boundary condition

$$(14) \qquad \mathcal{L}(z) = 0 \qquad \text{for } z \in C[0, T],$$

where $A_1(t, y(t), \lambda) = A(t, y(t) + \pi(t), \lambda)$, $F_1(t, y(t), \lambda) = F(t, y(t) + \pi(t), \lambda)$, and $f_1(t, y(t), \lambda) = \{A(t, y(t) + \pi(t), \lambda) - B(t)\}\pi(t)$.

We denote Z_y by the fundamental matrix solutions for the linear homogeneous system corresponding to (13) such that $Z_y(0) = I$. Put $V_y = I - Z_y(T)$, we have $\mathcal{L}(Z_y(\cdot)x_0) = V_y x_0$ for $x_0 \in \boldsymbol{R}^n$.

In the same argument as given in the proof of Theorem 1 it follows that for any $y \in C_{T,\varepsilon}$ and any $\lambda \in \Lambda_2$ there exists the inverse of V_y such that

$$\|V_y^{-1}\| \leqq 1/\{\rho(1-\rho)\}.$$

Moreover there exists one and only one solution $z_y \in C_{T,\varepsilon}$ for ((13), (14)) such that

$$z_y(t) = -V_y^{-1}[\mathcal{L}(q_y(\cdot))] + \int_0^t A_1(s, y(s), \lambda) z_y(s) ds$$

$$+\lambda\int_0^t F_1(s, y(s), \lambda)ds+\int_0^t f_1(s, y(s), \lambda)ds \qquad \text{for } t\in \boldsymbol{R},$$

where

$$q_y(t)=Z_y(t)\int_0^t Z_y^{-1}(s)\{\lambda F_1(s, y(s), \lambda)+f_1(s, y(s), \lambda)\}ds \qquad \text{for } t\in \boldsymbol{R}.$$

By the same argument used in the proof of Theorem 2, there exists at least one T-periodic solution $z(\cdot\,;\varepsilon,\lambda)\in C_{T,\varepsilon}$ for (13). Put $x(\cdot\,;\varepsilon,\lambda)=z(\cdot\,;\varepsilon,\lambda)+\pi(\cdot)$, we can see that there exists at least one T-periodic solution $x(\cdot\,;\varepsilon,\lambda)$ for (1) satisfying (12). Q.E.D.

When A, F satisfies the Lipschitz condition, respectively, we have the following theorem on the continuous dependence on λ of periodic solutions for (1).

Hypothesis 3. *There exists a positive constant L such that*

$$\|A(t, x_1, \lambda)-A(t, x_2, \lambda)\|\leqq L\,\|x_1-x_2\|$$

and that

$$\|F(t, x_1, \lambda)-F(t, x_2, \lambda)\|\leqq L\,\|x_1-x_2\|$$

for any $t\in[0, T]$, $x_i\in S_r$ $(i=1,2)$ *and* $\lambda\in\Lambda_2$.

Theorem 4. *Suppose that the assumption in Theorem* 3 *and Hypothesis* 3 *hold. If* $\lambda_2\leqq\lambda_2\Delta+M$ *and*

(15) $$2rLT\{K\exp(\delta)+r/(\lambda_2\Delta+M)\}<1,$$

then for any $\lambda\in\Lambda_2$ *there exists one and only one T-periodic solution* $x(\cdot\,;\lambda)$ *for* (1). *Moreover*

$$x(t\,;\lambda)\to\pi(t) \qquad \text{as } \lambda\to 0$$

uniformly in $t\in\boldsymbol{R}$.

Remark. From the second assertion of Theorem 4, the T-periodic solution for (1) is continuous in $\lambda\in\Lambda_2$.

Sketch of the proof of Theorem 4. Choose $\lambda\in\Lambda_2$. First, we consider the operator $\mathcal{F}: C_{T,r}\to C_{T,r}$ defined by $\mathcal{F}(y)=x_y$ for $y\in C_{T,r}$, where x_y is the T-periodic solution for (9) in $C_{T,r}$. It is easy to show that

$$[\mathcal{F}(y)](t)=-X_y(t)U_y^{-1}[\mathcal{L}(p_y(\cdot))]+p_y(t) \qquad \text{for } t\in\boldsymbol{R}.$$

We shall define k by the left-hand side of (15). It follows that

$$\|\mathcal{F}(y_1)-\mathcal{F}(y_2)\|_\infty\leqq k\,\|y_1-y_2\|_\infty \qquad \text{for } y_1, y_2\in C_{T,r}.$$

From $0<k<1$, the first assertion of the theorem holds.

Choose ε such that $0<\varepsilon<r-r_0$. Since the assumption of Theorem 3 holds, we can define the operator $\mathcal{G}: C_{T,\varepsilon}\to C_{T,\varepsilon}$ by $\mathcal{G}(y)=z_y$ for $y\in C_{T,\varepsilon}$, where z_y is the T-periodic solution for (13) in $C_{T,\varepsilon}$. In the same argument as the operator $\mathcal{F}$, $\mathcal{G}$ is a contraction. Therefore the second assertion holds. Q.E.D.

References

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