73. *Euler Number of Moduli Spaces of Instantons*

By Mikio FURUTA
Department of Mathematics, University of Tokyo

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1. Introduction. Let S^4 be the 4-dimensional sphere with the standard Riemannian metric, P_k a principal $SU(2)$-bundle over S^4 with $c_2(P_k)=k\ (k>0)$, and $P'_k=P_k/\{\pm 1\}$ the principal $SO(3)\ (=SU(2)/\{\pm 1\})$-bundle over S^4 with $p_1(P_k)=-4k$. We denote by M_k the moduli space of anti-instantons on P_k (or P'_k). It is known that M_k has a natural structure of $8k$-3 dimensional smooth manifold [3]. There are explicit descriptions of M_k [1] [2] [6], but not so much is known about the topology of M_k. S. K. Donaldson [6] and C. H. Taubes proved that M_k is connected. J. Hurtubise [10] proved that $\pi_1(M_k)=0$ if k is odd, and $\pi_1(M_k)=\boldsymbol{Z}/2$ if k is even.

It seems that some aspects of the topology of M_k is related to some profound properties of 4-dimensional smooth manifolds. In a sense, Donaldson's works in [5] and [7] about intersection forms of 4-manifolds are based on the fact that M_1 is diffeomorphic to open 5-disk.

The purpose of the present note is to announce our results about the Euler number of M_k.

2. Statement of the main results. Our first result is:

Theorem 1. *The Euler number $\chi(M_k)$ is equal to the number $d(k)$ of divisors of k.*

The orientation preserving isometry group $SO(5)$ of S^4 acts on M_k naturally [3]. Let $T=SO(2)\times SO(2)$ be the maximal torus of $SO(4)\ (\subset SO(5))$, and $M_k^T=\{[A]\in M_k\,;\, g[A]=[A] \text{ for any } g\in T\}$ the fixed point set. We reduce Theorem 1 to the following Theorem 2.

Theorem 2. *The number of the connected component of M_k^T is equal to $d(k)$, and each component is diffeomorphic to $\boldsymbol{R}$. Precisely, the number of lifts of T-action on P'_k is equal to $d(k)$, and our result is that for each lifted action, the moduli space of T-invariant anti-instanton on P'_k is diffeomorphic to $\boldsymbol{R}$.*

We can apply Theorem 2 to get some topological results [8].

3. Outline of the proof. Donaldson [6] showed that the moduli space of (framed) anti-instantons is identified with the moduli space of (framed) holomorphic vector bundle over $\boldsymbol{CP}^2=\boldsymbol{C}^2\cup\ell^\infty$ with rank$=2$ and trivial on the line ℓ^∞. To prove Theorem 2, we investigate T-equivariant holomorphic bundles over $\boldsymbol{CP}^2$. Here we regard T as the maximal torus of $SL(2,\boldsymbol{C})$. It could be possible to use the explicit description of M_k. To derive Theorem 1 from Theorem 2, we first show the following.

Lemma 3. *Let S^1 be a generic 1-dimensional connected subgroup of*

T. Then we have $M_k^{S^1}=M_k^T$. *For example, it suffices to take*

$$S^1=\{(t, t^p)\in T=SO(2)\times SO(2)\,;\, t\in SO(2)\}$$

for any fixed prime number p larger than k.

We give an outline of the proof of Lemma 3. If the class of an anti-instanton A in M_k is invariant under S^1-action, then it is shown that we can lift S^1-action on P'_k uniquely so that A is S^1-invariant. Although the lift depends on A, we can show that the dimension of the component of $M_k^{S^1}$ which contains the class of A is always equal to 1, using Lefschetz formula for equivariant Atiyah-Singer index theorem [4]. On the other hand, $CO(4)=\boldsymbol{R}_+SO(4)$ acts on M_k so that the $\boldsymbol{R}_+$-action is free, which is corresponding to the radial extention of $\boldsymbol{R}^4\cup\infty=S^4$. Therefore any component of $M_k^{S^1}$ is diffeomorphic to $\boldsymbol{R}$. Since any action of compact connected Lie group on $\boldsymbol{R}$ is trivial, any element of $M_k^{S^1}$ is invariant under T-action.

To get Theorem 1, we use the following lemma.

Lemma 4. *Let X be a (possibly open) manifold with* S^1*-action. Suppose that the rational cohomologies of X and* X^{S^1} *are finite dimensional. Then we have* $\chi(X)=\chi(X^{S^1})$.

Since M_k has a homotopy type of quasi-projective variety [5, 11], its rational cohomology is finite dimensional [9]. Thus we can apply Lemma 2 for $X=M_k$ to get Theorem 1.

The details of the proof will appear elsewhere.

References

[1] M. F. Atiyah: The geometry of Yang-Mills fields. Accademia Nazionale dei Lincei, Scuola Normale Supereore, Pisa (1979).

[2] M. F. Atiyah, V. G. Drinfeld, N. J. Hitchin, and Yu. I. Manin: Construction of instantons. Phys. Lett., **65**A, 185–187 (1978).

[3] M. F. Atiyah, N. J. Hitchin, and I. M. Singer: Self duality in four dimensional Riemannian geometry. Proc. Roy. Soc. London, Ser. A, **362**, 425–461 (1978).

[4] M. F. Atiyah and I. M. Singer: The index of elliptic operators III. Ann. of Math., **87**, 546–604 (1968).

[5] S. K. Donaldson: An application of gauge theory to four dimensional topology. J. Diff. Geom., **18**, 279–315 (1983).

[6] ——: Instantons and geometric invariant theory. Commun. Math. Phys., **93**, 453–460 (1984).

[7] ——: Connections, cohomology and the intersection form of 4-manifolds. J. Diff. Geom., **24**, 275–341 (1986).

[8] M. Furuta: Oriented $\boldsymbol{Z}/2$-homology cobordism classes represented by lens spaces (in preparation).

[9] A. Grothendieck: IHES, Paris, Pub. Math., **29**, 95–103 (1966).

[10] J. Hurtubise: Instantons and jumping lines. Commun. Math. Phys., **105**, 107–122 (1986).

[11] C. Okonek, M. Schneider, and H. Spindler: Vector Bundles on Complex Projective Spaces. Boston, Birkhäuser (1980).

[12] C. H. Taubes: Path connected Yang-Mills moduli spaces. J. Diff. Geom., **19**, 337–392 (1984).