40. *Isolated Singularities and Positive Solutions of Elliptic Equations in R^n*

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1. Introduction. We consider positive solutions to a second-order locally uniformly elliptic equation

(1) $$Pu \equiv [-\sum_{j,k=1}^{n} \partial_j(a_{jk}\partial_k) + \sum_{j=1}^{n} b_j\partial_j + c]u = 0$$

in a domain D of R^n, where $n \geqq 2$, $\partial_j = \partial/\partial x_j$, and a_{jk}, b_j, c are real-valued functions in $L_{\infty,\mathrm{loc}}(D)$, $L_{2p,\mathrm{loc}}(D)$, $L_{p,\mathrm{loc}}(D)$, respectively, for some $p > n/2$. A positive solution means a positive continuous function belonging locally to the Sobolev space of order 1 (i.e., in $H^1_{\mathrm{loc}}(D)$) and satisfying (1) in the weak sense (cf. [8]). We say that (P, D) is *subcritical* if for some y in D there exists a positive Green's function $G(\cdot, y)$ for P in D: $G \in L_{1,\mathrm{loc}}(D)$, $G > 0$, and $P_x G(x, y) = \delta(x-y)$ in the weak sense, where δ is Dirac's distribution (cf. [8]). Note that the subcriticality of (P, D) implies the existence of a positive solution to (1) in D (cf. [1], [2], and [3]).

The purpose of this note is to establish a relationship between positive solutions to an elliptic equation in a punctured ball and those to a corresponding equation in R^n. For isolated singularities of positive solutions, see [4, 5, 6, 7, 9]; as for positive solutions in R^n, see [1, 2, 3].

Let $\Omega = \{x \in R^n; 0 < |x| < R\}$ for some $R > 0$, and let P be an elliptic operator of the form as in (1) with $D = \{0 < |x| < R+1\}$. Choose a positive continuous function g on $\{0 < |x| \leqq R\}$ satisfying the equation $-\sum_{j,k} \partial_j(a_{jk}\partial_k g) + \sum_j b_j \partial_j g = 0$ in Ω. Define a generalized Kelvin transformation K by

(2) $$Ku(y) = u(y^*)/g(y^*), \qquad y^* = y|y|^{-2}.$$

Then the same argument as in the proof of Theorem 2 of [7] shows that if u is a solution of $Pu = 0$ in Ω, then Ku is a solution of the equation $P'v = 0$ in $E = \{y \in R^n; |y| > 1/R\}$, where

(3) $$\begin{aligned} P' &= -\sum_{i,l} \partial_i(a'_{il}\partial_l) + \sum_i b'_i\partial_i + c', \\ c'(y) &= h(y)|y|^{-4}c(y^*), \qquad h(y) = [g(y^*)|y|^{2-n}]^2, \\ b'_i(y) &= \sum_{j=1}^{n} h(y)|y|^{-2}b_j(y^*)(\delta_{ij} - 2y_i y_j/|y|^2), \\ a'_{il}(y) &= \sum_{j,k=1}^{n} h(y)a_{jk}(y^*)(\delta_{ij} - 2y_i y_j/|y|^2)(\delta_{kl} - 2y_k y_l/|y|^2). \end{aligned}$$

Here $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. For $K \geqq 0$, let P^K be an elliptic operator in R^n such that $P^K = P'$ in E and $P^K = -\Delta + K\chi_R$ in $R^n \setminus \bar{E}$ with χ_R being the characteristic function of the set $\{1/2R < |x| < 1/R\}$. Let

(4) $$H_+(P^K, R^n) = \{v \in H^1_{\mathrm{loc}}(R^n);\ P^K v = 0 \text{ and } v > 0 \text{ in } R^n\},$$

(5) $$H_+(P, \Omega, \{0\}) = \{u \in H^1_{\mathrm{loc}}(\Omega);\ u \text{ is continuous on } \bar{\Omega}\setminus\{0\} \text{ and vanishes on } \{|x| = R\},\ Pu = 0 \text{ and } u > 0 \text{ in } \Omega\}$$

be Fréchet spaces equipped with the metrics induced by L_∞-norms on compact subsets of R^n and Ω, respectively. Then we have

Theorem 1. *Suppose that (P, Ω) is subcritical. Then (P^K, R^n) is subcritical for some K; and for such a P^K there exists a bijection T from $H_+(P, \Omega, \{0\})$ to $H_+(P^K, R^n)$ which is linear, order-preserving, and continuous. Furthermore, there exists $C>1$ such that for any u in $H_+(P, \Omega, \{0\})$ and y with $|y|>2/R$*

$$(u/g)(y^*)\leqq(Tu)(y)\leqq C(u/g)(y^*). \tag{6}$$

Remark 1. Conversely, supposing that (P, R^n) is subcritical, we can obtain an analogous result.

Remark 2. Combining Theorem 1 and the results in [3] we can obtain several interesting results concerning isolated positive singularities.

2. Proof of Theorem 1.

Lemma 1. *(P^K, R^n) is subcritical for some K.*

Proof. We see that if there exists a positive solution to $P^K u=0$ in R^n for some K, then (P^{K+1}, R^n) is subcritical (cf. [1] and [3]). Therefore we have only to show that for some K the equation $P^K u=0$ in R^n has a positive solution. Suppose the contrary. Since the smallest real eigenvalue of the Dirichlet realization of P^K on $L_2(\{|x|<m\})$ must be negative for sufficiently large m (cf. the proof of Theorem 3.1, pp. 29 and 39 in [1]) and depends continuously on m, we obtain that for any $j=0, 1, \cdots$ there exist R_j and u_j such that $R^{-1}<R_0<R_1<\cdots$,

$$u_j \in H_0^1(\{|x|<R_j\}), \quad P^j u_j=0 \text{ and } u_j>0 \text{ in } \{|x|<R_j\},$$

and $u_j(x^0)=1$ for a fixed point x^0 with $R^{-1}<|x^0|<R_0$. By the Harnack inequality u_j converges as $j\to\infty$ to a positive solution u of $P'u=0$ in $D\equiv\{R^{-1}<|x|<\lim_{j\to\infty} R_j\}$. Since u_j are subsolutions of the equation $P^0 v=0$ in $\{|x|<R_0\}$, they are uniformly bounded in the ball. Elementary calculations together with the L_2-L_∞ estimate (see [Théorème 5.1, 8]) then show that $u_j(x)\to 0$ in $\{1/2R<|x|<1/R\}$ as $j\to\infty$. By Théorème 7.3 in [8], u vanishes continuously on $\{|x|=1/R\}$. Since (P, Ω) is subcritical, (P', D) is also subcritical. Thus, with G being the Green's function associated with the Dirichlet problem for P' in D, we obtain that $u(\cdot)\leqq\varepsilon G(\cdot, x^0)$ in D for any $\varepsilon>0$. Hence $u=0$ in D, which is a contradiction. Q.E.D.

Since (P', E) is subcritical, we see that for any $j>1/R$ and v in $H_+(P^K, R^n)$ there exists a unique solution w_j of the Dirichlet problem

$$P'w_j=0 \quad \text{in } \{1/R<|x|<j\},$$
$$w_j=u \text{ on } \{|x|=1/R\}, \qquad w_j=0 \text{ on } \{|x|=j\};$$

and that $Bv=\lim_{j\to\infty} w_j$ exists. Suggested by Nakai [4], we introduce an operator Π defined by: $\Pi v=v-Bv$. Then we have

Lemma 2. *Π is an isomorphism from $H_+(P^K, R^n)$ onto the Fréchet space $H_+(P', E, \{\infty\})$ of all positive solutions to $P'w=0$ in E vanishing continuously on ∂E. Furthermore, there exists a positive constant $C'<1$ such that for any v in $H_+(P^K, R^n)$*

$$C'v\leqq\Pi v\leqq v \quad \text{in } \{|x|>2/R\}.$$

Proof. Since (P^K, R^n) is subcritical, there is no non-trivial solution to

$P^K v=0$ in R^n which is equal to Bv_1-Bv_2 in E for some v_1 and v_2 belonging to $H_+(P^K, R^n)\cup\{0\}$. From this it follows that $\Pi v>0$ in E and Π is injective. Let us show the surjectivity of Π. Let $G=\lim_{j\to\infty} G_j$, where G_j is the Green's function associated with the Dirichlet problem for P^K in $\{|x|<j\}$. For any u in $H_+(P', E, \{\infty\})$, define v by

$$v(x)=U(x)-\int_{R^n} G(x, y)P^K U(y)dy,$$

where $U(x)=u(x)$ in E and $U(x)=0$ on $R^n\setminus E$. Then we see that $-P^K U$ is a measure with support in $\{|x|=1/R\}$ whose total measure is finite. Therefore v belongs to $H_+(P^K, R^n)$ and $\Pi v=u$. This proves the surjectivity. The last half of the lemma is easily shown by the Harnack inequality and the maximum principle. Q.E.D.

Completion of the proof of Theorem 1. From Lemmas 1 and 2 we get Theorem 1 by putting $T=\Pi^{-1}K$. Q.E.D.

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