28. On Persson's Theorem Concerning p-Nuclear Operators

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1. Let E, F be Banach spaces, p a real number such that $1 \leqq p < \infty$ and $1/p + 1/p' = 1$. We denote by $N_p(E, F)$ the set of all linear operators T from E into F which can be factorized as follows:

$$(*) \qquad E \xrightarrow{V} l^\infty \xrightarrow{D} l^p \xrightarrow{W} F$$

where V, W are bounded linear operators and $D = (\alpha_n)$ is a diagonal operator with $\sum_n |\alpha_n|^p < \infty$. The elements in $N_p(E, F)$ will be called p-nuclear operators or operators of type N_p. We also denote by $N^p(E, F)$ the set of all linear operators T from E into F which can be factorized as follows:

$$(**) \qquad E \xrightarrow{V} l^{p'} \xrightarrow{D} l^1 \xrightarrow{W} F$$

where V, W and D are of the same kind as above. The elements in $N^p(E, F)$ will be called operators of type N^p. For $p=1$ the two classes $N_p(E, F)$ and $N^p(E, F)$ are equal and coincide with the space of all nuclear operators from E into F. For $1 < p < \infty$ in general, neither $N_p(E, F) \subset N^p(E, F)$ nor the converse inclusion hold. In [3], Persson investigated some relation of these operators with p-integral and p-decomposable operators, and then proved that the inclusions $N^p(E, L^p) \subset N_p(E, L^p)$ and $N_p(L^{p'}, E) \subset N^p(L^{p'}, E)$ always hold for all Banach spaces E.

The purpose of this paper is to characterize Banach spaces E for which one of the following conditions holds:

(1) For each Banach space F, the inclusion $N^p(F, E) \subset N_p(F, E)$ holds.

(2) For each Banach space F, the inclusion $N_p(E, F) \subset N^p(E, F)$ holds.

We note that our results extend the works of Persson [3] and Kwapien [1]. As a consequence, we obtain that if E is of $S_{p'}$ type and F is of Q_p type in the sense of Kwapien [1], then the identity $N^p(E, F) = N_p(E, F)$ holds.

2. Main results. First we establish the relationship between p-nuclear operators and operators of type N^p. Throughout the paper, E denotes a Banach space with the dual E' and let p be $1 \leqq p < \infty$. In the following, $\{e_n\}$ denotes the sequence of canonical basis of $l^{p'}$, where $1/p + 1/p' = 1$.

Theorem 1. *Let T be a bounded linear operator from E into a Banach space F. Then we have the following.*

(1) *If T is p-nuclear, then T' (dual of T) is of type N^p.*

(2) *If T is of type N^p, then T' is p-nuclear.*

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Furthermore, if we assume that F is reflexive, then

(1′) *T is p-nuclear if and only if T' is of type N^p.*

(2′) *T is of type N^p if and only if T' is p-nuclear.*

Proof. Let us first remark that if S is a linear operator from $l^{p'}$ into a Banach space X such that $\sum_n \|Se_n\|^p < \infty$, then S is of type N^p and S' is p-nuclear. (1) If T is p-nuclear, then it has a factorization of $(*)$ (see Section 1). Evidently, we have $\sum_n \|D'e_n\|^p < \infty$. Hence D' is of type N^p, and so is T'. (2) If T is of type N^p, then it has a factorization of $(**)$. Since $\sum_n \|De_n\|^p < \infty$, D' is p-nuclear, and so is T'. Now suppose that F is reflexive. Then (1′) and (2′) follow from (1) and (2).

Let (Ω, Σ, μ) be a measure space. As usual, $L^p(\mu) = L(\Omega, \Sigma, \mu)$ denotes a Banach space of complex-valued measurable functions on Ω having p-integrable absolute value. Following Kwapien [1], we say that E is of S_p type (resp. Q_p type) if it is isomorphic to a subspace (resp. to a quotient) of some $L^p(\mu)$.

Theorem 2 (Takahashi and Okazaki [4]). *The following properties of a Banach space E are equivalent.*

(1) *E is of Q_p type.*

(2) *For each $T: l^{p'} \to E$, $\sum_n \|Te_n\|^p < \infty$ implies T is p-nuclear.*

Now we shall prove main theorems.

Theorem 3. *The following properties of a Banach space E are equivalent.*

(1) *E is of Q_p type.*

(2) *For each Banach space F, we have $N^p(F, E) \subset N_p(F, E)$.*

(3) *For some infinite dimensional space $L^{p'}(\mu)$, we have $N^p(L^{p'}, E) \subset N_p(L^{p'}, E)$.*

Proof. Let us first remark that every Banach space E has the properties (1), (2) and (3) for $p=1$. Hence we may assume that $1<p<\infty$. Suppose that (1) holds. To prove (2) let T be an operator of type N^p from F into E. Then T has a factorization of $(**)$ $V: F \to l^{p'}$, $D: l^{p'} \to l^1$ and $W: l^1 \to E$. Evidently, we have $\sum_n \|WDe_n\|^p < \infty$. Since E is of Q_p type, by Theorem 2 it follows that WD is p-nuclear, and so is T. Thus (2) holds. Obviously, (2) implies (3). It remains to prove that (3) implies (1). Suppose that (3) holds. Since every infinite dimensional space $L^{p'}$ contains a complemented subspace isomorphic to $l^{p'}$ (see [2]), the identity map: $l^{p'} \to l^{p'}$ is factorized by the bounded linear operators $V: l^{p'} \to L^{p'}$ and $W: L^{p'} \to l^{p'}$. To prove (1) let T be a linear operator from $l^{p'}$ into E such that $\sum_n \|Te_n\|^p < \infty$. Evidently, T is of type N^p and so is TW. Hence, by the assumption (3), TW is p-nuclear and so is $T = TWV$. By Theorem 2 it follows that E is of Q_p type and the proof is completed.

Theorem 4. *The following properties of a Banach space E are equivalent.*

(1) *E is of $S_{p'}$ type.*

(2) *For each Banach space F, we have $N_p(E, F) \subset N^p(E, F)$.*

(3) *For some infinite dimensional space* $L^p(\mu)$, *we have* $N_p(E, L^p) \subset N^p(E, L^p)$.

Proof. Let us first remark that every Banach space E has the properties (1), (2) and (3) for $p=1$. Hence we may assume that $1<p<\infty$. Suppose that (1) holds. To prove (2) let T be a p-nuclear operator from E into F. Then T has a factorization of (*) $V: E \to l^\infty$, $D: l^\infty \to l^p$ and $W: l^p \to F$. Evidently, DV is p-nuclear, and so $(DV)'$ is of type N^p (see Theorem 1). Since E' is of Q_p type, by Theorem 3 it follows that $(DV)'$ is p-nuclear. Hence, by Theorem 1 DV is of type N^p, and so is T. Thus (2) holds. Obviously, (2) implies (3). It remains to prove that (3) implies (1). Suppose that (3) holds. As in the proof of Theorem 3, the identity map: $l^p \to l^p$ is factorized by the bounded linear operators $V: l^p \to L^p$ and $W: L^p \to l^p$. To prove (1) it is enough to show that E' is of Q_p type. Let T be a linear operator from $l^{p'}$ into E' such that $\sum_n \|Te_n\|^p < \infty$. Then T' is clearly p-nuclear, and so is $T'J$, where J denotes the canonical isometry from E into E'' (bidual of E). Since $VT'J: E \to L^p$ is p-nuclear, by the assumption (3), it follows that $VT'J$ is of type N^p, and so is $T'J = WVT'J$. But this implies that $T=(T'J)'$ is p-nuclear (see Theorem 1). Thus the assertion follows from Theorem 2.

From Theorems 3 and 4 we have the following.

Corollary 1. *Suppose that E is of $S_{p'}$ type and F is of Q_p type. Then we have the identity* $N^p(E, F) = N_p(E, F)$.

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