20. On Automorphism Groups of Compact Riemann Surfaces of Genus 4

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Let X be a compact Riemann surface of genus $g \geqq 2$. A group AG of automorphisms of X (i.e., a subgroup of the group $\mathrm{Aut}(X)$ of all automorphisms of X) can be represented as a subgroup $R(X, AG)$ of $GL(g, \boldsymbol{C})$ as elements of AG operate in the g-dimensional module of abelian differentials on X. The purpose of this paper is to determine in case $g=4$ all subgroups of $GL(g, \boldsymbol{C})$ which are conjugate to some $R(X, AG)$ (for some X and some AG). (For the case $g=2, 3$ the same problem was already solved; [2] for the case $g=2$; the result for $g=3$ is not yet published.)

A more detailed account will be published elsewhere.

§1. **Preliminaries.** Let G be a finite subgroup of $GL(g, \boldsymbol{C})$, and H a non-trivial cyclic subgroup of G. Define two sets $CY(G)$ and $CY(G; H)$ by

$CY(G) := \{K ; K$ is a non-trivial cyclic subgroup of $G\}$,

$CY(G; H) := \{K \in CY(G) ; K$ contains strictly a subgroup H of $G\}$.

We say that G satisfies the condition (F) if for every element A of G, $r(A) := 2-(\mathrm{Tr}(A)+\mathrm{Tr}(A^{-1}))$ is a non-negative integer. Further we define as follows:

(1) $\quad r(H) := 2-(\mathrm{Tr}(A)+\mathrm{Tr}(A^{-1}))$, where $H=\langle A\rangle$.

(2) $\quad r_*(H) := r(H) - \sum_K r_*(K)$ (defined by descending condition)

where K ranges over the set $CY(G; H)$.

(3) $\quad l(H) := r_*(H)/[N_G(H) : H]$, $l(I) := 0$, where I is the trivial group.

(4) $$g_0(G) := (1/\sharp G) \sum_{A \in G} \mathrm{Tr}(A).$$

Then we have the following relation [2]:

(RH) $$2g-2 = \sharp G(2g_0-2) + \sharp G \sum_i l(H_i)(1-(1/n_i)).$$

Here $\{H_i\}$ is a complete set of representatives of G-conjugacy classes of $CY(G)$ and $n_i := \sharp H_i$. We put further $\sharp G = n$.

We say that a finite subgroup G of $GL(g, \boldsymbol{C})$ satisfies (RH_+) if G satisfies (F) and if $l(H)$ is a non-negative integer for any H of $CY(G)$. Then put $\mathrm{RH}(G) := [g_0, n; n_1, \cdots, n_1, \cdots, n_s, \cdots, n_s]$, where n_i appears $l(H_i)$-times $(1 \leqq i \leqq s)$.

We say that a finite subgroup G of $GL(g, \boldsymbol{C})$ satisfies the condition (E) if the following conditions are satisfied:

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(i) For each element $M(\sharp M=n>1)$ of G there exist integers $\nu_1, \cdots, \nu_r$ such that

$$\mathrm{Tr}(M)=1+\sum_{i=1}^{r}\zeta_n^{\nu_i}/(1-\zeta_n^{\nu_i}), \qquad \zeta_n=\exp(2\pi i/n),$$

where $1\leqq\nu_i\leqq n-1$, $(\nu_i, n)=1$ and $r=2-(\mathrm{Tr}(M)+\mathrm{Tr}(M^{-1}))\geqq 0$.

(ii) For any A, B of G such that $B=A^k$, $k|m$ $(k\neq m)$, $m=\sharp A$, the trace formula for B does not conflict with the one for A.

We say that G satisfies the condition (K) if it satisfies (RH_+) and (E). We know that $R(X, AG)$ satisfies the condition (K) [2, 3].

Notations. For the sake of simplicity we put as follows:

$$\bar{A}=\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&-1&0\\0&0&0&-1\end{bmatrix} \quad \bar{B}=\begin{bmatrix}0&-1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix} \quad \bar{C}=\begin{bmatrix}0&0&1&0\\1&0&0&0\\0&0&0&1\\0&1&0&0\end{bmatrix} \quad \bar{D}=\begin{bmatrix}0&1&0&0\\0&0&1&0\\1&0&0&0\\0&0&0&1\end{bmatrix}$$

$$\bar{E}=\begin{bmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{bmatrix} \quad U=\begin{bmatrix}a&b&c&d\\b&d&a&c\\c&a&d&b\\d&c&b&a\end{bmatrix}, \quad \begin{matrix}b-c=a\\b+c=-1\\bc=1/5\\d=-a\end{matrix}; \quad D(a,b,c,d)=\begin{bmatrix}a&0&0&0\\0&b&0&0\\0&0&c&0\\0&0&0&d\end{bmatrix};$$

$$L=\frac{1}{\sqrt{2}}\begin{bmatrix}\omega\zeta^3&\omega\zeta^7&0&0\\\omega\zeta^5&\omega\zeta^5&0&0\\0&0&\zeta^3&\zeta^7\\0&0&\zeta^5&\zeta^5\end{bmatrix}, \quad \zeta=\zeta_8, \quad \omega=\zeta_3.$$

§2. Maximal subgroups. First, we consider all possible n which satisfy RH(G) in §1. Next, for each possible n considered above we construct all possible groups of order n which satisfy the condition (E). Thus we have maximal subgroups of $GL(4, \boldsymbol{C})$ among them which satisfy the condition (K).

(A-1) (1) $G(120)=\langle D(\zeta,\zeta^2,\zeta^3,\zeta^4), \bar{C}, U\rangle$, $\zeta=\zeta_5$.
(2) $G(8\times 9)=\langle D(\omega,\omega,\omega,\omega^2), \bar{B}, \bar{D}\rangle$, $\zeta=\zeta_3$.
(3) $G(9\times 8)=\langle D(1,1,\omega,\omega^2), \bar{E}, \bar{A}\rangle$.
(4) $G(4\times 9)=\langle D(\zeta,\zeta,\zeta^2,\zeta^3), D(\omega,\omega^2,\omega,\omega), \bar{A}\rangle$, $\zeta=\zeta_6$.
(5) $G(18)=\langle D(\omega,\omega^2,1,1), D(\omega,\omega,\omega,\omega^2), \bar{A}\rangle$.
(6) $G(2\times 6)=\langle D(\zeta,\zeta^5,-1,1), \bar{A}\rangle$, $\zeta=\zeta_6$.
(7) $G(6\times 2)=\langle D(\zeta,\zeta^2,\zeta^4,\zeta^5), \bar{E}\rangle$, $\zeta=\zeta_6$.
(8) $G(2\times 2\times 3)=\langle D(-1,-1,1,-1), D(1,-1,-1,1), D(\omega,\omega,\omega,\omega^2)\rangle$.
(9) $G(8,8)=\langle D(i,-i,-1,1), \bar{A}\rangle$, $i=\sqrt{-1}$.

(A-2) (1) $G(15)=\langle D(\zeta,\zeta^2,\zeta^8,\zeta^{11})\rangle$, $\zeta=\zeta_{15}$.
(2) $G(12)=\langle D(\zeta,\zeta^7,\zeta^2,\zeta^3)\rangle$, $\zeta=\zeta_{12}$.
(3) $G(10)=\langle D(\zeta,\zeta^2,\zeta^4,\zeta^7)\rangle$, $\zeta=\zeta_{10}$.

(B) (1) $H(40)=\langle D(\zeta,\zeta^2,\zeta^3,\zeta^4), \bar{E}\rangle$, $\zeta=\zeta_{10}$.
(2) $H(32)=\langle D(\zeta,\zeta^3,\zeta^5,\zeta^7), \bar{E}\rangle$, $\zeta=\zeta_{16}$.
(3) $H(24)=\langle D(i,-i,i,-i), L\rangle$.
(4) $H(18)=\langle D(\zeta,\zeta^3,\zeta^5,\zeta^7)\rangle$, $\zeta=\zeta_{18}$.

All other groups are contained up to $GL(4, C)$-conjugacy in the group listed above and there are 74 groups in all (including the trivial one).

It remains to show that these groups are conjugate to some $R(X, AG)$

for some X of genus 4 and some AG.

§3. **The expressions of curves.** Let X be a non-hyperelliptic curve of genus 4. X is contained in a unique irreducible quadric surface defined by $Q=0$ and X is the complete intersection of $Q=0$ with an irreducible cubic surface defined by $F=0$ [1]. If $Q=0$ and $F=0$ are A-invariant, then A comes from an automorphism of the curve $X=\{(Q=0)\cap(F=0)\}$. The converse of this statement is true as was shown by F. Momose. Hence we can give curves for the groups in (A-1) and (A-2) by this method. However for (A-2) we can also obtain by an elementary method.

(A-1) Non-cyclic case:

(1) $Q: X_1X_4+X_2X_3=0,\quad F: X_1^2X_3-X_2^2X_1-X_3^2X_1+X_4^2X_2=0.$

(2) $Q: X_1^2+X_2^2+X_3^2=0,\quad F: X_4^3-X_1X_2X_3=0.$

(3) $Q: X_1X_2+X_3X_4=0,\quad F: X_1^3-X_2^3+X_3^3-X_4^3=0.$

(4) $Q: X_3^2+X_1X_4=0,\quad F: X_1^3-X_2^3+X_4^3=0.$

(5) $Q: X_3^2+X_1X_2=0,\quad F: X_3^3+X_4^3+X_1X_2X_3=0.$

(6) $Q: X_3^2+X_4^2+X_1X_2=0,\quad F: X_1^3-X_2^3+X_3^3+X_4^2X_3+cX_1X_2X_3=0.$

(7) $Q: X_1X_2+X_3X_4=0,\quad F: X_1^3+X_4^3+X_2^2X_4+X_3^2X_1=0.$

(8) $Q: X_1^2+X_2^2+X_3^2=0,\quad F: X_1^3+X_4^3+X_2^2X_1+X_3^2X_1=0.$

(9) $Q: X_1^2+X_2^2+X_3X_4=0,\quad F: X_3^3+X_1^2X_4+X_2^2X_4+X_4^2X_3+cX_1X_2X_3=0.$

Here c in (6) and (9) are arbitrary constants which make the equations irreducible.

(A-2) Cyclic case: We can easily obtain the equations [4].

(1) $y^{15}=x^2(x-1)^3.$

(2) $y^{12}=x(x-1)^2.$

(3) $y^{10}=x(x-1)^3.$

(B) The curves in this block are hyperelliptic [4].

(1) $y^2=x^{10}-1.$

(2) $y^2=x(x^8-1).$

(3) $y^2=x(x^4-1)(x^4+2\sqrt{-3}\,x^2+1).$

(4) $y^2=x(x^9-1).$

§4. **Main Theorem.** From §2 and §3 we are able to get the following

Theorem. *Let G be a finite subgroup of $GL(4, \boldsymbol{C})$. Then the following two conditions are equivalent.*

(1) *There is a compact Riemann surface X of genus 4 and an automorphism group of X such that $R(X, AG)$ is $GL(4, \boldsymbol{C})$-conjugate to G.*

(2) *G satisfies the condition* (K).

Remark. Corresponding theorems hold also for genera 2 and 3. However, this is not the case for genus 5 as was shown by F. Momose by constructing a counter-example.

References

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