2. Instability of Periodic Solutions of Some Evolution Equations Governed by Time-Dependent Subdifferential Operators

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Let H be a Hilbert space with norm $|\cdot|$, and $\Phi(H)$ be the set of all proper lower semicontinuous convex functions from H into $(-\infty, \infty]$. Given a T-periodic mapping $t \mapsto \phi^t$ from $\boldsymbol{R}$ into $\Phi(H)$, and a T-periodic function f in $L^2_{\mathrm{loc}}(\boldsymbol{R}; H)$ (i.e. $\phi^{t+T}=\phi^t$ for $t \in \boldsymbol{R}$, and $f(t+T)=f(t)$ for a.e. $t \in \boldsymbol{R}$), we consider the equation

$$\text{(E)} \qquad u'(t)+\partial\phi^t(u(t)) \ni f(t), \qquad t \in J,$$

where J is an interval in $\boldsymbol{R}$, $u'(t)=(d/dt)u(t)$ and $\partial\phi^t$ is the subdifferential of ϕ^t. For related studies on (E) we refer to [2, 4, 7, 8, 11, 12, 13].

In [3], Baillon and Haraux treated the time-independent case of ϕ^t, i.e. $\phi^t \equiv \phi$, and proved that any solution on $J=[t_0, \infty)$ is asymptotically T-periodic in the weak topology of H and the difference of any two T-periodic solutions is a constant vector on $\boldsymbol{R}$. Subsequently, Haraux [5] and Ishii [6] discussed the equation from the same viewpoint as in [3], when $\phi^t \equiv \phi$ and f is almost periodic on $\boldsymbol{R}$. In this paper we shall show by a simple example in 3-dimensional space that the equation with the time-dependent ϕ^t is essentially different in nature from that with the time-independent $\phi^t \equiv \phi$.

1. A flow in 3-dimensional space. We take 3-dimensional space $\boldsymbol{R}^3$ as H, and denote by $x=(x_1, x_2, x_3)$ a generic point in $\boldsymbol{R}^3$. Now, for each $t \in \boldsymbol{R}$ and $\theta \in [0, \infty)$, let us consider the operator $R_\theta(t)$ from the x_1x_2-plane $X_0=\{(x_1, x_2, 0); x_1, x_2 \in \boldsymbol{R}\}$ into $\boldsymbol{R}^3$ which is defined as follows:

$$(1) \qquad R_\theta(t)x=(x_1(t), x_2(t), x_3(t)), \qquad x \in X_0,$$

where $x_1(t)=r(\cos\theta\cos(\theta-\alpha)+\sin\theta\sin(\theta-\alpha)\cos t)$, $x_2(t)=r(\sin\theta\cos(\theta-\alpha) -\cos\theta\sin(\theta-\alpha)\cos t)$, $x_3(t)=r\sin t\sin(\theta-\alpha)$ and $x=r(\cos\alpha, \sin\alpha, 0)$, $r=|x|$, $0 \leqq \alpha < 2\pi$. The operation $x \mapsto R_\theta(t)x$ geometrically means the rotation of x around the line l_θ: $-x_1\tan\theta+x_2=x_3=0$ in t-degree. From the definition of $R_\theta(t)$ we immediately see that

(2) $R_\theta(t)$ is linear and isometric for any $t \in \boldsymbol{R}$ and $\theta \in [0, \pi)$, and

(3) $R_\theta(t)$ is a C^∞-function of t for any $x \in X_0$ and $\theta \in [0, \pi)$.

For the moment we fix a number θ with $0<\theta<\pi$. For each $t \in \boldsymbol{R}$ we define the operator $S(t)$ from X_0 into $\boldsymbol{R}^3$ by

$$(4) \qquad S(t)=S_0(t-2n\pi)[S_0(2\pi)]^n \qquad \text{for } t \in [2n\pi, 2(n+1)\pi),\ n \in \boldsymbol{Z},$$

where

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$$S_0(t)=\begin{cases}R_0(t) & \text{for } 0\leqq t\leqq \pi,\\ R_\theta(t-\pi)R_0(\pi) & \text{for } \pi<t\leqq 2\pi,\end{cases}$$

and

$$[S_0(2\pi)]^0=\text{the identity on } X_0,\ [S_0(2\pi)]^n=\{[S_0(2\pi)]^{-1}\}^{-n} \qquad \text{for } n<0.$$

Also, we denote by $X(t)$ the image of X_0 under $S(t)$, i.e. $X(t)=S(t)X_0$. Clearly, if $t\in[2n\pi, (2n+1)\pi]$ (resp. $t\in[(2n+1)\pi, 2(n+1)\pi]$), then $X(t)$ is the plane in $\boldsymbol{R}^3$ which is given by rotating X_0 around the line $l_0: x_2=x_3=0$ (resp. l_θ) in $(t-2n\pi)$-degree (resp. $(t-\pi-2n\pi)$-degree). Moreover,

(5) $\qquad X(n\pi)=X_0,\ X(t+2n\pi)=X(t) \qquad$ for all $n\in\boldsymbol{Z}$ and $t\in\boldsymbol{R}$.

Proposition 1. *Let $0<\theta<\pi$, and $S(t)$ be as above. Then we have:*

(S1) *For each $t\in\boldsymbol{R}$, $S(t)$ is a linear isometric operator from X_0 onto $X(t)$.*

(S2) *$S(0)$ is the identity on X_0.*

(S3) *$S(2n\pi)x=r(\cos(\alpha+2n\theta), \sin(\alpha+2n\theta), 0)$ for any $x=r(\cos\alpha, \sin\alpha, 0)$, $r=|x|$, $0\leqq\alpha<2\pi$.*

(S4) *$S(t)x=S(t-2n\pi)S(2n\pi)x$ for any $t\in\boldsymbol{R}$, $n\in\boldsymbol{Z}$ and $x\in X_0$.*

(S5) (i) *For each $x\in X_0$, $S(\cdot)x$ is a Lipschitz continuous function on $\boldsymbol{R}$ with $|x|$ as a Lipschitz constant and belongs to $C^\infty(\boldsymbol{R}\backslash\Delta;\boldsymbol{R}^3)$, where $\Delta=\{n\pi; n\in\boldsymbol{Z}\}$.* (ii) *The right* (resp. *left*) *derivative $(d^+/dt)S(t)x$* (resp. *$(d^-/dt)S(t)x$*) *exists for every $t\in\boldsymbol{R}$ and $x\in X_0$, and $(d^\pm/dt)S(t)x\in X(t)^\perp=$ the orthogonal complement of $X(t)$ in $\boldsymbol{R}^3$ for every $t\in\boldsymbol{R}$ and $x\in X_0$.* (iii) *For $x\in X_0$, $(d^\pm/dt)S(t)x=0$ for some $t\in\boldsymbol{R}$ if and only if $x=0$.*

(S6) (i) *If $x\in X_0$, $x\neq 0$ and $S(\cdot)x$ has a period $T>0$, then $T=2n\pi$ for some $n\in\boldsymbol{N}$.* (ii) *Let $x\in X_0$, $x\neq 0$ and $n\in\boldsymbol{N}$. Then $S(\cdot)x$ is $2n\pi$-periodic if and only if $\theta=k\pi/n$ for some $k\in\boldsymbol{N}$.* (iii) *If $x\in X_0$, $x\neq 0$ and θ/π is irrational, then $S(\cdot)x$ has no period.*

(S7) *For each $x\in X_0$, $S(\cdot)x$ is almost periodic on $\boldsymbol{R}$.*

Proof. Properties (S1)–(S5) immediately follow from (1)–(5). Now we prove (i) of (S6). Let $T>0$, $x\in X_0$ and $x\neq 0$, and suppose that

(6) $\qquad S(t+T)x=S(t)x \qquad$ for all $t\in\boldsymbol{R}$.

Put $T=T_0+2n\pi$ with $0\leqq T_0<2\pi$ and $n\in\boldsymbol{N}$ or $n=0$. Then it suffices to show that $T_0=0$ and $n\in\boldsymbol{N}$. For this purpose we show that any of the following three cases (α), (β), (γ) never occurs: (α) $0<T_0<\pi$. (β) $T_0=\pi$. (γ) $\pi<T_0<2\pi$. First assume (α) holds. Then we note that (6) with $t=0$ and (4) yield

(7) $\qquad x=S(T)x=R_0(T_0)[S_0(2\pi)]^n x.$

Hence $x\in X_0\cap X(T_0)=l_0$. Since $R_0(T_0)$ is the identity on l_0, it follows from (7) that $x=[S_0(2\pi)]^n x$. From this and the equality (6) with $t=-T_0$ we see that

$$x=[S_0(2\pi)]^n x=S(-T_0+T)x=S(-T_0)x=R_\theta(-T_0)x.$$

Therefore $x\in X_0\cap X(-T_0)=l_\theta$. Since $l_0\cap l_\theta=\{0\}$, we have $x=0$. This is a contradiction. Similarly, under (β) or (γ) we get a contradiction. Thus (i) of (S6) holds. Property (ii) of (S6) is easily derived from (S3) and (6), and subsequently (iii) of (S6) holds. Finally, we show (S7) by using the following Bochner's criterion for the almost periodicity (cf. Amerio-Prouse

[1]): A function $f \in C(\boldsymbol{R}; \boldsymbol{R}^3)$ is almost periodic on $\boldsymbol{R}$ if and only if for any sequence $\{s_k\}$ in $\boldsymbol{R}$, there exists a subsequence $\{s_{k_j}\}$ of $\{s_k\}$ such that $f(t+s_{k_j})$ converges in $\boldsymbol{R}^3$ uniformly in $t \in \boldsymbol{R}$. Now, let $\{s_k\}$ be any sequence in $\boldsymbol{R}$ and $x = r(\cos\alpha, \sin\alpha, 0)$. Then, putting

$$s_k = 2n_k\pi + \tau_k,\ n_k \in \boldsymbol{Z},\ \tau_k \in [0, 2\pi),\ 2n_k\theta = 2m_k\pi + \theta_k,\ m_k \in \boldsymbol{Z},\ \theta_k \in [0, 2\pi),$$

we obtain from (S3) and (S4) that

$$S(t+s_k)x = S(t+s_k-2n_k\pi)S(2n_k\pi)x = S(t+\tau_k)x_k,$$

where $x_k = r(\cos(\alpha+2n_k\theta), \sin(\alpha+2n_k\theta), 0) = r(\cos(\alpha+\theta_k), \sin(\alpha+\theta_k), 0)$. Here, extract a subsequence $\{k_j\}$ of $\{k\}$ so that $\tau_{k_j} \to \tau_0 \in [0, 2\pi]$ and $\theta_{k_j} \to \theta_0 \in [0, 2\pi]$. Then $x_{k_j} \to x_0 = r(\cos(\alpha+\theta_0), \sin(\alpha+\theta_0), 0) \in X_0$ in $\boldsymbol{R}^3$. Therefore, by (S1) and (i) of (S5),

$$\begin{aligned} |S(t+s_{k_j})x - S(t+\tau_0)x_0| &\leqq |S(t+\tau_{k_j})x_{k_j} - S(t+\tau_{k_j})x_0| + |S(t+\tau_{k_j})x_0 - S(t+\tau_0)x_0| \\ &\leqq |x_{k_j} - x_0| + |x_0| \cdot |\tau_{k_j} - \tau_0|. \end{aligned}$$

This shows that $S(t+s_{k_j})x$ converges to $S(t+\tau_0)x_0$ in $\boldsymbol{R}^3$ uniformly in $t \in \boldsymbol{R}$.

2. An example. With the same notations as in the previous section, for each $t \in \boldsymbol{R}$ we put

$$\phi^t(x) = \begin{cases} 0 & \text{if } x \in X(t), \\ \infty & \text{if } x \in \boldsymbol{R}^3 \setminus X(t). \end{cases}$$

Clearly $\phi^t \in \Phi(\boldsymbol{R}^3)$ with $D(\partial\phi^t) = X(t)$, and $\partial\phi^t(x) = X(t)^\perp$ for any $x \in X(t)$. By (5), the mapping $t \mapsto \phi^t$ is 2π-periodic on $\boldsymbol{R}$. Moreover, (S5) implies that for every $x \in X_0$, $u(t) = S(t)x$ gives a solution to the equation

$$u'(t) + \partial\phi^t(u(t)) \ni 0, \qquad t \in \boldsymbol{R}. \tag{8}$$

Denoting by $\mathscr{P}_T$ the set of all T-periodic solutions of (8), we obtain immediately the following propositions from the facts in the previous section.

Proposition 2. *Suppose that $\theta = k\pi/n \in (0, \pi)$ for some $k, n \in \boldsymbol{N}$. Then every solution u of* (8) *belongs to $\mathscr{P}_{2n\pi}$, and for any $u, v \in \mathscr{P}_{2n\pi}$ with $u \neq v$ the difference $u - v$ is not constant on $\boldsymbol{R}$. Moreover we have $\mathscr{P}_{2\pi} = \{0\}$.*

Proposition 3. *Suppose that θ/π is an irrational number in* $(0, 1)$. *Then $\mathscr{P}_T = \{0\}$ for every $T > 0$, and hence, if u is a solution of* (8) *and $u \not\equiv 0$ on $\boldsymbol{R}$, then u is not T-periodic on $\boldsymbol{R}$ for any $T > 0$. Moreover, every solution u of* (8) *is an almost periodic function on $\boldsymbol{R}$ such that $|u(t)| = |u(0)|$ for all $t \in \boldsymbol{R}$.*

Remarks. In general, for a solution u to (E), $-u'(t)$ does not coincide with the minimal section $(\partial\phi^t(u(t)) - f(t))^0$ of $\partial\phi^t(u(t)) - f(t)$. This is one of the reasons why the behavior of solutions is quite different from that of the equation with the time-independent $\phi^t \equiv \phi$.

The detail discussion on the behavior of solutions to (E) will be made in the authors' forthcoming papers [9, 10].

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