63. *On the Spaces of Self Homotopy Equivalences of Certain CW Complexes*

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1. Introduction. Let X be a connected CW complex with base point which is a vertex of X. And let $G(X)$ and $G_0(X)$ be the space of self homotopy equivalences of X with the compact open topology and the space (subspace of $G(X)$) of self homotopy equivalences of X preserving the base point, respectively. When X is an Eilenberg-MacLane complex $K(\pi, n)$, the weak homotopy type of $G(X)$ and $G_0(X)$ are completely determined by R. Thom [4] and D. H. Gottlieb [1], but it seems that little is known about the homotopy type of $G(X)$ and $G_0(X)$.

2. Results. Now, let X and Y be connected locally finite CW complexes with base points. Then there exists the following homeomorphisms (see [3]),

$$(X\times Y)^{X\times Y}\cong X^{X\times Y}\times Y^{X\times Y}\cong (X^X)^Y\times (Y^Y)^X,$$
$$(X\times Y)_0^{X\times Y}\cong X_0^{X\times Y}\times Y_0^{X\times Y}=(X^X, X_0^X)^{(Y,y_0)}\times (Y^Y, Y_0^Y)^{(X,x_0)},$$

where Z_0^K denotes the space of maps of K to Z preserving the base points with the compact open topology, $(Z, Z')^{(K,L)}$ denotes the space of maps of (K, L) to (Z, Z') and $(Z, Z')^{(K,L)}$ is regarded as a subspace of Z^K. Under these correspondences we have the following two theorems.

Theorem 1. *Let X and Y be connected locally finite CW complexes with base points. For given $n>0$, assume that $\pi_i(X)=0$ for every $i>n$ and $\pi_i(Y)=0$ for every $i\leq n$. Then we have*

$$G(X\times Y)=G(X)^Y\times G(Y)^X,$$
$$G_0(X\times Y)=(G(X), G_0(X))^{(Y,y_0)}\times (G(Y), G_0(Y))^{(X,x_0)}.$$

Theorem 2. *For given $n>0$, let X be a connected locally finite CW complex with base point whose dimension is not greater than n and let Y be an n-connected locally finite CW complex with base point. Then the same formulas on $G(X\times Y)$ and $G_0(X\times Y)$ as in Theorem 1 hold.*

These theorems are obtained by considering the induced homomorphisms of homotopy groups of self map of $(X\times Y, (x_0, y_0))$.

Let X be a connected locally finite CW complex with base point. Then every arcwise connected component of $G(X)$ has the same homotopy type. The same fact holds for $G_0(X)$. More generally, we have the following

Proposition 1. *Let X be a homotopy associative H-space with unit e. Suppose for each element x of X there exists an element x' of X such that $x \cdot x'$ and $x' \cdot x$ both are contained in the arcwise connected component of e. Then, every arcwise connected component of X has the same homotopy type.*

Let us consider a relation between $G(X)$ and $G_0(X)$ of a connected CW complex X with base point. We have

Proposition 2. *Let X be a connected CW complex with base point which is also an H-space. Then $G(X)$ has the same weak homotopy type as $X \times G_0(X)$.*

Now, by performing a proof within the category of compactly generated spaces and maps along the argument used in the proofs of Theorems 1 and 2, we can obtain the following

Theorem 3. *For given $n>0$, let X be a connected CW complex with base point and let Y be an n-connected CW complex with base point. Assume that $\dim X \leqq n$ or $\pi_i(X)=0$ for every $i>n$. Then the following holds*

$$G(X \times Y) \underset{w}{\simeq} G(X) \times G(Y) \times G(X)_0^Y \times G(Y)_0^X,$$

$$G_0(X \times Y) \underset{w}{\simeq} G_0(X) \times G_0(Y) \times G(X)_0^Y \times G(Y)_0^X,$$

where $\underset{w}{\simeq}$ means to have the same weak homotopy type.

By setting $X=K(\pi, n)$ in Theorem 3, we have

Corollary 1. *Let X be an n-connected CW complex with base point. Then we have*

$$G_0(K(\pi, n) \times X) \underset{w}{\simeq} \operatorname{Aut}(\pi) \times G_0(X) \times G(X)_0^{K(\pi,n)}.$$

Corollary 2. *Let X be a simply connected finite CW complex with base point. Then we have*

$$G(S^1 \times X) \simeq O(2) \times G(X) \times \Omega G(X),$$

$$G_0(S^1 \times X) \simeq Z_2 \times G_0(X) \times \Omega G(X),$$

where $O(2)$ is the orthogonal group of degree 2 and $\Omega G(X)$ is the loop space of $G(X)$ based at the identity map id_X of X.

3. Applications. Suppose X and Y are connected CW complexes with base points. Let us denote by $\varepsilon(X)$ and $\varepsilon(Y)$ the group of based homotopy classes of self homotopy equivalences of X and Y respectively. Then in the following we can define an action of the direct product $\varepsilon(X) \times \varepsilon(Y)$ of $\varepsilon(X)$ and $\varepsilon(Y)$ on the group $[X, G(Y)]_0$ whose multiplication is induced by the H-structure in $G(Y)$. Let k be an element of $G_0(Y)$ and let $G_i(Y)$ be the arcwise connected component of $G(Y)$ containing id_Y. We define a self map $\tilde{k}$ of $G_i(Y)$ by using the multiplication in $G(Y)$ as follows:

$$\tilde{k}(\alpha)=k^{-1} \cdot \alpha \cdot k \qquad (\alpha \in G_i(Y)),$$

where k^{-1} is a fixed element of $G_0(Y)$ which represents the inverse element of $[k]$. Let $[\bar{f}]$ be an element of $[X, G(Y)]_0=[X, G_i(Y)]_0$, then we have a well-defined action of $\varepsilon(X)\times\varepsilon(Y)$ on $[X, G_i(Y)]_0$ as follows:

$$([h], [k])^*[\bar{f}]=[\tilde{k}\circ\bar{f}\circ h].$$

If $\pi_j(X)=0$ for every $j>n$ and Y be n-connected, then we define a correspondence λ of $\varepsilon(X\times Y)$ to $(\varepsilon(X)\times\varepsilon(Y))\otimes[X, G(Y)]_0$ which is a semi-direct product of the groups $\varepsilon(X)\times\varepsilon(Y)$ and $[X, G(Y)]_0$ defined by the action introduced above. As a bi-product of Theorems 1 and 3, we have the following

Theorem 4. *For given $n>0$, let X be a connected CW complex with base point such that $\pi_i(X)=0$ for every $i>n$ and let Y be an n-connected CW complex with base point. Then λ is an isomorphism of $\varepsilon(X\times Y)$ onto the semi-direct product group $(\varepsilon(X)\times\varepsilon(Y))\otimes[Y, G(X)]_0$ defined by the action introduced above.*

As a special case of this theorem, we have a generalization of the theorem of S. Sasao and Y. Ando [2] as follows.

Corollary. *Let X be an n-connected CW complex with base point. Then we have an isomorphism λ:*

$$\varepsilon(K(\pi, n)\times X)\longrightarrow(\mathrm{Aut}\,(\pi)\times\varepsilon(X))\otimes[K(\pi, n), G(X)]_0,$$

where the group on the right hand is a semi-direct product of two groups $\mathrm{Aut}\,(\pi)\times\varepsilon(X)$ and $[K(\pi, n), G(X)]_0$.

Finally, by observing that the direct product $\varepsilon(X)\times\varepsilon(Y)$ of the groups $\varepsilon(X)$ and $\varepsilon(Y)$ is acting on the group $[Y, G(X)]_0=[Y, G_i(X)]_0$, we have the following result.

Theorem 5. *For given $n>0$, let X be a connected CW complex of $\dim X\leqq n$ with base point and let Y be an n-connected CW complex with base point. Suppose that $[X, G(Y)]_0$ is trivial, then λ is an isomorphism of $\varepsilon(X\times Y)$ onto the semi-direct product group $(\varepsilon(X)\times\varepsilon(Y))\otimes[Y, G(X)]_0$ defined by the action introduced above.*

As a special case of Theorem 5, we have

Corollary. *For given $n>0$, let X be a connected CW complex of $\dim X\leqq n$ with base point. Then we have the following isomorphism*

$$\lambda: \varepsilon(X\times K(\pi, n+1))\longrightarrow(\varepsilon(X)\times\mathrm{Aut}\,(\pi))\otimes[K(\pi, n+1), G(X)]_0.$$

Details will appear elsewhere.

References

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