29. The Spectrum of the Laplacian and Smooth Deformation of the Riemannian Metric

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§ 1. **Introduction.** Let M be an n-dimensional compact connected C^∞ manifold (with or without boundary ∂M). Every Riemannian metric g of M determines the Laplace-Beltrami operator Δ_g. We consider the eigenvalue problem for $-\Delta_g$ (under Dirichlet condition);

$$\text{(1.1)} \qquad \begin{cases}(-\Delta_g-\lambda)u(x)=0 & \\ u(x)=0 & (\text{if } \partial M\neq\phi).\end{cases}$$

Let $0\leq\lambda_0\leq\lambda_1\leq\lambda_2\cdots$ be the eigenvalues of the problem (1.1). These are determined by the metric g. The totality of Riemannian metrics of class C^∞ which differ from a fixed metric g_0 only on an open set $U\subset M$ forms a separable Fréchet manifold B.

Theorem A. *If* $\dim M=n\geq 2$, *then there exists a residual subset* $\Gamma\subset B$ *such that all eigenspaces of* $-\Delta_g$ *are one dimensional for any* $g\in\Gamma$.

We call a subset Γ residual if it is a countable intersection of open dense subsets. Since a residual set is dense and a second category by virtue of Baire's theorem, Theorem A implies that for almost all $g\in B$ the eigenvalues of problem (1.1) are all simple.

In our proof we follow the idea of Uhlenbeck [6], who has already obtained the similar result in the case that those metrics are of class C^k $(n+3\leq k<+\infty)$. But the first transversality theorem of her can not be applied to our case, since B is not a Banach manifold. We need the following Fréchet manifold version of the transversality theorem.

Theorem B. *Let* E, F *and* G *be strong ILH manifolds of class* C^r. *Assume that* E *and* F *are separable. Let the mapping* $f: E\times F\to G$ *be a* C^r*-strong ILH mapping satisfying the following conditions;*

(a) *For every* $u\in E\times F$, *every* $k\in N(d)$,

$$\text{(1.2)} \qquad \|(Df^k)_u\delta u\|_k\geq C_u\,\|\delta u\|_k-D_u^k\,\|\delta u\|_{k-1},$$

where $\delta u\in T_u(E\times F)$, C_u *and* D_u^k *are positive constants and* C_u *is independent of* k.

(b) *There exists* $p\in G$ *such that* p *is a regular value of* f. *(That is for any* $u\in f^{-1}(p)$ *the Fréchet derivative* $(Df)_u$ *is onto.)*

(c) *For every* $b\in F$, $f_b=f(\ ,b): E\to G$ *is a strong ILH Fredholm mapping with index* $<r$.

Then the set $\{b\in F\,;\, p$ *is a regular value of* $f_b\}$ *is residual in* F.

§ 2. **Transversality theorem.** Let $N(d)$ be the set of all integers k satisfying $k \geq d$. We call a system $\{E, E^k, k \in N(d)\}$ a Sobolev chain, if the following conditions hold;

(A) every E^k is a Hilbert space,

(B) E^{k+1} is continuously, linearly and densely imbedded in E^k,

(C) E is an intersection of all E^k with inverse (projective) limit topology.

Let $\{E, E^k, k \in N(d)\}$ and $\{F, F^k, k \in N(d)\}$ be two Sobolev chains, $U \subset E^d$ and $U' \subset F^d$ be open neighbourhoods of $x_0 \in E$ and $y_0 \in F$, respectively. A mapping $f: U \cap E \to U' \cap F$ is called a strong *ILH* mapping of class C^r $(r \geq 2)$, if f satisfies the following conditions;

(i) f can be extended to a C^r-mapping $f^k: U \cap E^k \to U' \cap F^k$ for every $k \in N(d)$,

(ii) for any $x \in U \cap E$, there exists an E^d-neighbourhood $W_x \subset U$ such that for every $u \in W_x \cap E$ and $v, v_1, v_2 \in E$, we have

$$\|(Df^k)_u v\|_k \leq C_x(\|u-x\|_k \|v\|_d + \|v\|_k) + P_x^k(\|u-x\|_{k-1}) \|v\|_{k-1} \tag{2.1}$$

and

$$\begin{aligned} \|(D^2 f^k)_u(v_1, v_2)\|_k \leq C_x(&\|u-x\|_k \|v_1\|_d \|v_2\|_d + \|v_1\|_k \|v_2\|_d + \|v_1\|_d \|v_2\|_k) \\ &+ P_x^k(\|u-x\|_{k-1}) \|v_1\|_{k-1} \|v_2\|_{k-1}, \end{aligned} \tag{2.2}$$

where C_x is a positive constant independent of k and P_x^k is a polynomial with positive coefficients depending on k.

Theorem 2.1 (Implicit function theorem, Omori [3]). *Let $f: U \cap E \to U' \cap F$ be a C^r-strong ILH mapping with $f(x_0) = y_0$;*

(I) *$(Df^k)_{x_0}: E^k \to F^k$ is an isomorphism for every $k \in N(d)$,*

(II) *for every $k \in N(d)$, we have*

$$\|(Df^k)_{x_0} v\|_k \geq C \|v\|_k - D_k \|v\|_{k-1}, \tag{2.3}$$

where C and D_k are positive constants and C is independent of k. Then there exist open neighbourhoods $V \subset E^d$ and $V' \subset F^d$ of x_0 and y_0, respectively, such that f is a C^r-diffeomorphism from $V \cap E$ into $V' \cap F$ and f^{-1} is also a C^r-strong ILH mapping satisfying the inequality (2.3).

By virtue of the Theorem 2.1, we can consider manifolds modeled on Sobolev chains and we call such manifolds strong *ILH* manifolds. *ILH* means *i*nverse *l*imit *H*ilbert. (See Omori [3].)

A C^r-strong *ILH* mapping $f: U \cap E \to U' \cap F$ is called a Fredholm mapping if $(Df^k)_x: E^k \to F^k$ is a Fredholm operator for every $k \in N(d)$, every $x \in U \cap E$ and the index of $(Df^k)_x$ is independent of k.

Theorem 2.2. *Let $f: U \cap E \to U' \cap F$ be a C^r-strong ILH Fredholm mapping with $r > \max$ (index of f, 1). Assume that*

$$\|(Df^k)_x v\|_k \geq C_x \|v\|_k - D_x^k \|v\|_{k-1}, \tag{2.4}$$

for every $x \in U \cap E$, $v \in E$ and $k \in N(d)$, where C_x and D_x^k are positive constants and C_x is independent of k. Then the regular values of f form a residual set in F.

Theorem B follows from Theorem 2.2.

§3. **Sketch of the proof of Theorem A.** We denote by H^k the $\boldsymbol{R}$-algebras of k-th order Sobolev functions of M into $\boldsymbol{R}$ and we set $H = C^\infty(M)$ as the inverse (projective) limit of H^k. We set $S^k = \{u \in H^{k+2}, \|u\|_{L^2(M)} = 1$, and $u(x) = 0$, $x \in \partial M\}$ and $S = \varprojlim_k S^k$. H and S are strong ILH manifolds of class C^∞.

We define by $H^k_{\bar{U}}(M, T^*M \circledS T^*M)$ the totality of H^k-sections of $T^*M \circledS T^*M$ with support in $\bar{U}$, where $T^*M \circledS T^*M$ is the symmetric tensor product of cotangent bundle T^*M and U is an open subset of M. We fix $m > n/2 + 2$ and choose an open neighbourhood $V \subset H^m_{\bar{U}}(M, T^*M \circledS T^*M)$ of 0-section such that every g in $g_0 + V$ is a C^2-Riemannian metric of M. We set $B^k = (g_0 + V) \cap H^{k+m}(M, T^*M \circledS T^*M)$ and $B = \varprojlim_k B^k$. B is also a strong ILH manifold of class C^∞.

Let Δ_g be the Laplace-Beltrami operator with respect to a Riemannian metric $g \in B$. We consider the mapping $f: S \times \boldsymbol{R} \times B \to H$ given by $f(u, \lambda, g) = (-\Delta_g - \lambda)u$, where $u \in S$, $\lambda \in \boldsymbol{R}$ and $g \in B$. We can easily prove the following propositions.

Proposition 3.1. *f is a strong ILH mapping of class C^∞. For every $g \in B$, the mapping $f_g = f(\ ,\ , g): S \times \boldsymbol{R} \to H$ is a Fredholm mapping with index $= 0$.*

Proposition 3.2. *f satisfies the inequality* (1.2).

The following proposition is due to Uhlenbeck [6].

Proposition 3.3. *$-\Delta_g$ has only one dimensional eigenspaces if and only if $0 \in H$ is a regular value of f_g.*

Just as in Proposition 2.10 in Uhlenbeck [6], we can prove

Proposition 3.4. *$0 \in H$ is a regular value of f.*

We can apply Theorem B to f replacing E by $S \times \boldsymbol{R}$, F by B, G by H and p by $0 \in H$. Then we can prove Theorem A.

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