66. *Studies on Holonomic Quantum Fields. IX*

By Michio JIMBO

Research Institute of Mathematical Sciences, Kyoto University

(Communicated by Kôsaku YOSIDA, M. J. A., Nov. 13, 1978)

In this note we shall give a symplectic version of the 2-dimensional operator theory, previously expounded in the orthogonal case [2], [5], [6]. Of particular interest is the neutral theory discussed in § 4. Corresponding to the bose field $\varphi^F(a)$ [1], there arises a strongly interacting fermi field $\varphi^B(a)={}^t(\varphi^B_+(a), \varphi^B_-(a))$. These two fields $\varphi^F(a)$ and $\varphi^B(a)$ are shown to share the same S-matrix in common, and their τ-functions are related to each other through simple formulas (34), (36), (38)–(39) (cf. IV–(49) [2]).

We remark that the 1-dimensional Riemann-Hilbert problem [4], [8] is also dealt with in the symplectic framework.

We follow the notations used throughout this series [1]–[6].

The author wishes to express his heartiest thanks to Prof. M. Sato and Dr. T. Miwa for many discussions and valuable suggestions.

1. Let W be an N-dimensional complex vector space equipped with a skew-symmetric inner product $\langle\ ,\ \rangle$. Let $A(W)$ be the algebra generated by W with the defining relation $ww'-w'w=\langle w, w'\rangle$. Denote by $S(W)$ the symmetric tensor algebra over W. As in the orthogonal case [3], [7], the norm map

$$\mathrm{Nr}: A(W) \xrightarrow{\sim} S(W) \tag{1}$$

and the expectation value $\langle a\rangle$ of $a \in A(W)$ are defined analogously, by specifying a bilinear form $(w, w')\to\langle ww'\rangle$ on W such that $\langle ww'\rangle - \langle w'w\rangle = \langle w, w'\rangle$ $(w, w' \in W)$.

Now let $v_1, \cdots, v_N$ be a basis of W, and set $K=(\langle v_\mu v_\nu\rangle)$, $H=(\langle v_\mu, v_\nu\rangle)=K-{}^tK$. Consider an element g of the form

$$\mathrm{Nr}\,(g)=\langle g\rangle e^{\rho/2}, \qquad \rho=\textstyle\sum_{\mu,\nu=1}^N R_{\mu\nu}v_\mu v_\nu = vR^tv \tag{2}$$

with $v=(v_1, \cdots, v_N)$. Contrary to the orthogonal case, $e^{\rho/2}$ no longer belongs to $S(W)$. So we let $R_{\mu\nu}=R_{\nu\mu} \in t\cdot \boldsymbol{C}[[t]]$, and regard g (resp. $e^{\rho/2}$) as an element of $A(W)[[t]]$ (resp. $S(W)[[t]]$), the formal power series ring with coefficients in $A(W)$ (resp. $S(W)$). The norm map (1) is uniquely extended there. (This formulation is due to T. Miwa.) Most of the formulas in the orthogonal case are valid for g of the form (2), if we replace ${}^tK$ by $-{}^tK$. We tabulate below formulas corresponding to (1.5.5)–(1.5.6), (1.5.7)–(1.5.8) and (1.4.6)–(1.4.7) of [7].

$$\mathrm{Nr}\,(wg)=\left(\textstyle\sum_{\mu,\nu=1}^N v_\mu(1+R^tK)_{\mu\nu}c_\nu\right)\cdot\langle g\rangle e^{\rho/2} \tag{3}$$

$$\mathrm{Nr}\,(gw)=(\textstyle\sum_{\mu,\nu=1}^{N} v_\mu(1+RK)_{\mu\nu}c_\nu)\cdot\langle g\rangle e^{\rho/2}$$

where $w=\sum_{\mu=1}^{N} v_\mu c_\mu$.

$$(4)\qquad g\cdot v_\nu=(\textstyle\sum_{\mu=1}^{N} v_\mu T_{\mu\nu})\cdot g$$

$$(5)\qquad R(K\overline{\oplus}{}^tKT)=T-1,\qquad T=(T_{\mu\nu}).$$

$$\langle g\rangle^2=gg^*\cdot\det(1+KR)$$

where $*$ is defined in [1]. Let $\Lambda=(\lambda_{\mu\nu})={}^t\Lambda$ be such that $\lambda_{\nu\nu}=1$ ($\nu=1,\cdots,n$) and define $W(\Lambda)=\oplus_{\nu=1}^{n} W^{(\nu)}$ analogously as in [7]. Then

$$(6)\qquad \mathrm{Nr}\,(g^{(1)}\cdots g^{(n)})=\langle g^{(1)}\cdots g^{(n)}\rangle e^{\hat\rho/2}$$

$$\langle g^{(1)}\cdots g^{(n)}\rangle=\langle g^{(1)}\rangle\cdots\langle g^{(n)}\rangle\det(1-A(\Lambda)R)^{-1/2}$$

$$\hat R=R(1-A(\Lambda)R)^{-1},\qquad \hat\rho=\hat v\hat R{}^t\hat v$$

where $\mathrm{Nr}(g^{(\nu)})=\langle g^{(\nu)}\rangle e^{v^{(\nu)}R^{(\nu)}{}^tv^{(\nu)}/2}$, $\hat v=(v^{(1)},\cdots,v^{(n)})$,

$$R=\begin{pmatrix}R^{(1)}&&\\&\ddots&\\&&R^{(n)}\end{pmatrix}\quad\text{and}\quad A(\Lambda)=\begin{bmatrix} & \lambda_{12}K & \cdots & \lambda_{1n}K\\ \lambda_{12}{}^tK & & & \vdots\\ \vdots & & \ddots & \lambda_{n-1\,n}K\\ \lambda_{1n}{}^tK & \cdots & \lambda_{n-1\,n}{}^tK & \end{bmatrix}.$$

Notice the exponent $-1/2$ of the determinant in (6), which differ from the orthogonal case by sign.

2. Let $\phi(u)$, $\phi^*(u)$ denote the creation ($u<0$)-annihilation ($u>0$) operators of complex free bose field. Their commutation relations and expectation values read

$$(7)\qquad \begin{pmatrix}[\phi(u),\phi(u')] & [\phi(u),\phi^*(u')]\\ [\phi^*(u),\phi(u')] & [\phi^*(u),\phi^*(u')]\end{pmatrix}=\begin{pmatrix} & 1\\ 1 & \end{pmatrix}2\pi|u|\delta(u+u')$$

$$(8)\qquad \begin{pmatrix}\langle\phi(u)\phi(u')\rangle & \langle\phi(u)\phi^*(u')\rangle\\ \langle\phi^*(u)\phi(u')\rangle & \langle\phi^*(u)\phi^*(u')\rangle\end{pmatrix}=\begin{pmatrix} & 1\\ 1 & \end{pmatrix}2\pi u_+\delta(u+u').$$

For $l\in C$ we set

$$(9)\qquad \phi_l(x)=\int \underline{du}(0+iu)^l e^{-im(x^-u+x^+u^{-1})}\phi(u),$$

$$\phi_l^*(x)=\int \underline{du}(0+iu)^l e^{-im(x^-u+x^+u^{-1})}\phi^*(u).$$

In the case $l=0$ we write (9) simply as $\phi(x)$ and $\phi^*(x)$, respectively. Set further

$$(10)\qquad \rho_B(a\,;l)=2\iint \underline{du}\,\underline{du'}R_B(u,u'\,;l)e^{-im(a^-(u+u')+a^+(u^{-1}+u'^{-1}))}\phi(u)\phi^*(u'),$$

$$R_B(u,u'\,;l)=-2\sin\pi l\Big(\frac{u-i0}{u'-i0}\Big)^{-l+1/2}\frac{\sqrt{u-i0}\sqrt{u'-i0}}{u+u'-i0},$$

and define $\varphi_B(a\,;l)$, $\varphi_{l'}^B(a\,;l)$ and $\varphi_{l'}^{B*}(a\,;l)$ as follows.

$$(11)\qquad \mathrm{Nr}\,(\varphi_B(a\,;l))=\exp\,(\rho_B(a\,;l)/2)$$

$$\mathrm{Nr}\,(\varphi_{l'}^B(a\,;l))=\phi_{l'}(a)\exp\,(\rho_B(a\,;l)/2)$$

$$\mathrm{Nr}\,(\varphi_{l'}^{B*}(a\,;l))=\phi_{l'}^*(a)\exp\,(\rho_B(a\,;l)/2).$$

Notice that if $l=0$, $l'=0$ (11) reduce to 1, $\phi(a)$ and $\phi^*(a)$, respectively. The local expansions corresponding to VII–(6), (7) are valid. Assuming $l \notin \boldsymbol{Z}$ we have

$$(12)\qquad \begin{aligned}\mathrm{Nr}\,(\phi(x)\varphi_B(a\,;l)) &= \textstyle\sum_{j=0}^{\infty} \mathrm{Nr}\,(\varphi^{B}_{-l+1+j}(a\,;l))\cdot v_{-l+1+j}[a]\\ &\quad + \textstyle\sum_{j=0}^{\infty} \mathrm{Nr}\,(\varphi^{B}_{-l-j}(a\,;l))\cdot v^{*}_{l+j}[a]\\ \mathrm{Nr}\,(\phi^*(x)\varphi_B(a\,;l)) &= \textstyle\sum_{j=0}^{\infty} \mathrm{Nr}\,(\varphi^{B*}_{l+j}(a\,;l))\cdot v_{l+j}[a]\\ &\quad + \textstyle\sum_{j=0}^{\infty} \mathrm{Nr}\,(\varphi^{B*}_{l-1-j}(a\,;l))\cdot v^{*}_{-l+1+j}[a],\end{aligned}$$

$$(13)\qquad \begin{aligned}\mathrm{Nr}\,(\phi(x)\varphi^{B*}_{l'}(a\,;l)) &= \frac{1}{2\sin \pi l'}\mathrm{Nr}\,(\varphi_B(a\,;l))\cdot(v_{-l'}[a]-v^{*}_{l'}[a])\\ &\quad + \mathrm{Nr}\,(\phi(x)\varphi_B(a\,;l))\cdot\phi^{*}_{l'}(a),\\ \mathrm{Nr}\,(\phi^*(x)\varphi^{B}_{l'}(a\,;l)) &= \frac{1}{2\sin \pi l'}\mathrm{Nr}\,(\varphi_B(a\,;l))\cdot(v_{-l'}[a]-v^{*}_{l'}[a])\\ &\quad + \mathrm{Nr}\,(\phi^*(x)\varphi_B(a\,;l))\cdot\phi_{l'}(a).\end{aligned}$$

Here we have set $v_l[a]=v_l(-(x-a)^-+i0,\,(x-a)^+-i0)$, etc. In (12) and (13), if the order of product is reversed as $\varphi_B(a\,;l)\phi(x)$ and so forth, we are only to replace the boundary values $\mp(x-a)^{\mp}\pm i0$ by $\mp(x-a)^{\mp}\mp i0$ (compare the case of fermion VII–(7) where a change in sign should be incorporated). We note in particular the relations

$$(14)\qquad \begin{aligned}\phi(x)\varphi^{B*}_{l}(a\,;l) &= \frac{1}{2\sin \pi l}\Big(\varphi_B(a\,;l)\cdot v_{-l}[a]-m^{-1}\frac{\partial}{\partial(-a^-)}\varphi_B(a\,;l)\\ &\qquad \cdot v_{-l+1}[a]+\cdots\Big)+(\text{terms involving } v^{*}_{l+j}[a],\ j\geqq 0),\\ \phi^*(x)\varphi^{B}_{-l}(a\,;l) &= \frac{1}{2\sin \pi l}\Big(\varphi_B(a\,;l)\cdot v^{*}_{-l}[a]-m^{-1}\frac{\partial}{\partial a^+}\varphi_B(a\,;l)\\ &\qquad \cdot v^{*}_{-l+1}[a]+\cdots\Big)+(\text{terms involving } v_{l+j}[a],\ j\geqq 0).\end{aligned}$$

As a result of (12), (13) the operators (11) enjoy the following commutation relations with the free field $\phi(x)$, $\phi^*(x)$ for spacelike separation of x and a:

$$(15)\qquad \varphi(a\,;l)\phi(x)=\begin{cases}\phi(x)\varphi(a\,;l) & (x^+>a^+,\ x^-<a^-)\\ e^{2\pi i l}\phi(x)\varphi(a\,;l) & (x^+<a^+,\ x^->a^-)\end{cases}$$

for $\varphi(a\,;l)=\varphi_B(a\,;l)$ or $\varphi^{B*}_{l'}(a\,;l)$ with $l'\equiv l \bmod \boldsymbol{Z}$,

$$(16)\qquad \varphi(a\,;l)\phi^*(x)=\begin{cases}\phi^*(x)\varphi(a\,;l) & (x^+>a^+,\ x^-<a^-)\\ e^{-2\pi i l}\phi^*(x)\varphi(a\,;l) & (x^+<a^+,\ x^->a^-)\end{cases}$$

for $\varphi(a\,;l)=\varphi_B(a\,;l)$ or $\varphi^{B}_{l'}(a\,;l)$ with $l'\equiv -l \bmod \boldsymbol{Z}$.

3. Making use of the operators in § 2 we now introduce our wave functions in the Minkowski space-time X^{Min}. For $\nu=1,\cdots,n$ we set

$$(17)\qquad \begin{aligned}\tau_{Bn}v_0(x^*,x\,;L) &= \pi\langle\phi^*(x^*)\varphi_B(a_1\,;l_1)\cdots\varphi_B(a_n\,;l_n)\phi(x)\rangle\\ \tau_{Bn}v_\nu(x,L) &= 2\sin \pi l_\nu\langle\varphi_B(a_1\,;l_1)\cdots\varphi^{B*}_{l_\nu}(a_\nu\,;l_\nu)\cdots\varphi_B(a_n\,;l_n)\phi(x)\rangle\end{aligned}$$

where $\tau_{Bn}=\tau_{Bn}(L)=\langle\varphi_B(a_1\,;l_1)\cdots\varphi_B(a_n\,;l_n)\rangle$ denotes the τ-function. These functions (17) are analytically prolongable to the subdomain of $(X^{C})^{n+2}$ (and in particular that of $(X^{\mathrm{Euc}})^{n+2}$) defined by $\mathrm{Im}\,(x^*-a_\nu)^\pm<0$, $\mathrm{Im}\,(a_\mu-a_\nu)^\pm<0$ $(1\leqq\mu<\nu\leqq n)$ and $\mathrm{Im}\,(x-a_\nu)^\pm>0$, to result in the canonical basis $v_0(L)$, $v_\nu(L)$ in VIII–§§ 1, 2, respectively. In the sequel the vacuum expectation values (17), τ_{Bn}, etc. are often confused with their Euclidean continuations. From (14) we have

$$\alpha_{\nu\nu}(L) = -m^{-1}\partial_{a_\nu} \log \tau_B(L) \qquad (\nu=1, \cdots, n) \tag{18}$$

in the notation of VIII. In view of the characterization of these Euclidean wave functions (cf. VIII–(5)), we see that the following relations hold between the "fermi" and "bose" wave functions:

$$\begin{aligned} (19)\quad & i\langle \psi_-^*(x^*)\varphi_F(a_1\,;l_1)\cdots\varphi_F(a_n\,;l_n)\psi_+(x)\rangle/\tau_{Fn}(L) \\ &= \langle \phi^*(x^*)\varphi_B(a_1\,;l_1+1/2)\cdots\varphi_B(a_n\,;l_n+1/2)\phi(x)\rangle/\tau_{Bn}(L+1/2) \\ (20)\quad & i\langle \varphi_F(a_1\,;l_1)\cdots\varphi_{l_\nu}^{F*}(a_\nu\,;l_\nu)\cdots\varphi_F(a_n\,;l_n)\psi_+(x)\rangle/\tau_{Fn}(L) \\ &= \langle \varphi_B(a_1\,;l_1+1/2)\cdots\varphi_{l_\nu+1/2}^{B*}(a_\nu\,;l_\nu+1/2) \\ &\quad \cdots\varphi_B(a_n\,;l_n+1/2)\phi(x)\rangle/\tau_{Bn}(L+1/2). \end{aligned}$$

On the other hand, from (18) and VIII–(21) the "fermi" and "bose" τ-functions $\tau_{Fn}(L)=\langle\varphi_F(a_1\,;l_1)\cdots\varphi_F(a_n\,;l_n)\rangle$ and $\tau_{Bn}(L+1/2)$ are themselves related through

$$d\log\tau_{Bn}(L+1/2) = -d\log\tau_{Fn}(L) = -\omega \tag{21}$$

where ω denotes the 1-form VIII–(20). For instance if $n=2$ we have

$$\omega = \left(t\left(\left(\frac{d\psi}{dt}\right)^2 - \sinh^2\psi\right) - t^{-1}l^2\tanh^2\psi\right)dt/2 \tag{22}$$

where $t=2m\,|a_1-a_2|$, $l=l_1-l_2$, and $\psi=\psi(t)$ satisfies

$$\frac{d^2\psi}{dt^2} + \frac{1}{t}\frac{d\psi}{dt} = \frac{1}{2}\sinh 2\psi + \left(\frac{l}{t}\right)^2 \tanh\psi\cdot\operatorname{sech}^2\psi. \tag{23}$$

Equation (23) is converted into a Painlevé equation of the fifth kind by the substitution $y=\tanh^2\psi$, $x=t^2$. By the boundary conditions τ_{Bn}, $\tau_{Fn}\to 1$ ($|a_\mu-a_\nu|\to\infty$ for all $\mu\neq\nu$) (21) implies further that

$$\tau_{Bn}(L+1/2)\cdot\tau_{Fn}(L) = 1. \tag{24}$$

Introduction of the parameter $\Lambda=(\lambda_{\mu\nu})$ is carried out similarly as in VII [5]. Let $\phi^{(\mu)}(u)$, $\phi^{*(\mu)}(u)$ ($\mu=1, \cdots, n$) denote copies of $\phi(u)$, $\phi^*(u)$. The inner product $\langle\ ,\ \rangle_\Lambda$ and the vacuum expectation value $\langle\ \rangle_\Lambda$ of μ-th and ν-th copies are set equal to $\lambda_{\mu\nu}=\lambda_{\nu\mu}$ times the original ones, where we assume $\lambda_{\nu\nu}=1$ ($\nu=1, \cdots, n$) as before. Define $\phi_l^{(\nu)}(x)$, $\phi_l^{*(\nu)}(x)$, $\varphi_B^{(\nu)}(a\,;l)$, $\varphi_{l'}^{B(\nu)}(a\,;l)$ and $\varphi_{l'}^{B*(\nu)}(a\,;l)$ by using $\phi^{(\nu)}(u)$, $\phi^{*(\nu)}(u)$ in place of $\phi(u)$, $\phi^*(u)$ respectively. We have then

$$\begin{aligned} (25)\quad & i\langle \psi_-^{*(\mu)}(x^*)\varphi_F^{(1)}(a_1\,;l_1)\cdots\varphi_F^{(n)}(a_n\,;l_n)\psi_+^{(\nu)}(x)\rangle_\Lambda/\tau_{Fn}(L\,;\Lambda) \\ &= \langle \phi^{*(\mu)}(x^*)\varphi_B^{(1)}(a_1\,;l_1+1/2) \\ &\quad \cdots\varphi_B^{(n)}(a_n\,;l_n+1/2)\phi^{(\nu)}(x)\rangle_\Lambda/\tau_{Bn}(L+1/2\,;\Lambda) \\ (26)\quad & i\langle \varphi_F^{(1)}(a_1\,;l_1)\cdots\varphi_{l_\mu}^{F*(\mu)}(a_\mu\,;l_\mu)\cdots\varphi_F^{(n)}(a_n\,;l_n)\psi_+^{(\nu)}(x)\rangle_\Lambda/\tau_{Fn}(L\,;\Lambda) \\ &= \langle \varphi_B^{(1)}(a_1\,;l_1+1/2)\cdots\varphi_{l_\mu+1/2}^{F*(\mu)}(a_\mu\,;l_\mu+1/2) \\ &\quad \cdots\varphi_B^{(n)}(a_n\,;l_n+1/2)\phi^{(\nu)}(x)\rangle_\Lambda/\tau_{Bn}(L+1/2\,;\Lambda) \end{aligned}$$

where $\tau_{Fn}(L\,;\Lambda)=\langle\varphi_F^{(1)}(a_1\,;l_1)\cdots\varphi_F^{(n)}(a_n\,;l_n)\rangle_\Lambda$ and $\tau_{Bn}(L+1/2\,;\Lambda)$ $=\langle\varphi_B^{(1)}(a_1\,;l_1+1/2)\cdots\varphi_B^{(n)}(a_n\,;l_n+1/2)\rangle_\Lambda$ are related through

$$\tau_{Bn}(L+1/2\,;\Lambda)\tau_{Fn}(L\,;\Lambda) = 1. \tag{27}$$

4. In the special case $l=1/2$, it is possible to construct operator

theory based on neutral bose field $\phi(u)$. The field $\varphi_B(a)$ is introduced in I [1] along with $\varphi_F(a)$ and $\varphi^F(a)$. We set further

(28) $$\mathrm{Nr}\,(\varphi^B_{\pm}(a))=\Big(\int \underline{du}\sqrt{0+iu}^{\pm 1}e^{-im(a^- u+a^+ u^{-1})}\phi(u)\Big)\cdot \mathrm{Nr}\,(\varphi_B(a)).$$

From the definition, $\varphi^B(a)={}^t(\varphi^B_+(a),\varphi^B_-(a))$ transforms as a spinor. It is shown that if a and a' are mutually spacelike, then

(29) $$[\varphi_B(a),\varphi_B(a')]=0$$
$$[\varphi^B_\varepsilon(a),\varphi^B_{\varepsilon'}(a')]_+=0 \qquad (\varepsilon,\varepsilon'=\pm).$$

Moreover the asymptotic fields for $\varphi^B_\varepsilon(a)$ $(\varepsilon=\pm)$

(30) $$\phi^B_{\varepsilon\pm}(u)=\epsilon(u)\lim_{t\to\pm\infty}\frac{i}{2}\int_{x^0=t}dx^1(e^{im(x^- u+x^+ u^{-1})}\partial_0\varphi^B_\varepsilon(x)$$
$$-\varphi^B_\varepsilon(x)\cdot\partial_0 e^{im(x^- u+x^+ u^{-1})})$$

exist and are calculated exactly. We find

(31) $$\phi^B_{\varepsilon\pm}(u)=(0+iu)^{\varepsilon(1/2)}\psi^B_\pm(u)$$

where

(32) $$\mathrm{Nr}\,(\psi^B_\pm(u))=\phi(u)\cdot\exp\Big(-2\int_0^\infty \underline{du'}\theta(\pm(|u|-u'))\phi^+(u')\phi(u')\Big)$$

satisfy the canonical anti-commutation relations $[\psi^B_\pm(u),\psi^B_\pm(u')]_+ =2\pi|u|\,\delta(u+u')$ for free fermion (cf. I–(2)[1]). As in the case of $\varphi^F(a)$, the asymptotic state vectors are related to the auxiliary ones through

(33) $$\langle vac|\psi^B_\pm(u_1)\cdots\psi^B_\pm(u_k)=\prod_{i<j}\epsilon(\pm(u_i-u_j))\cdot\langle vac|\phi(u_1)\cdots\phi(u_k)$$
$$\psi^{B\dagger}_\pm(u_k)\cdots\psi^{B\dagger}_\pm(u_1)|vac\rangle=\prod_{i<j}\epsilon(\pm(u_i-u_j))\cdot\phi^\dagger(u_k)\cdots\phi^\dagger(u_1)|vac\rangle$$

where $\psi^{B\dagger}_\pm(u)=\psi^B_\pm(-u)$.

To sum up, $\varphi^B(a)={}^t(\varphi^B_+(a),\varphi^B_-(a))$ is a fermion field satisfying Lorentz covariance, microcausality and asymptotic completeness, and its S-matrix is given by $S=(-)^{N(N-1)/2}$ where N denotes the total particle-number operator.

Just as in the complex case, the relation with the Euclidean deformation theory enables us to express the τ-functions for $\varphi_B(a)$ and $\varphi^B(a)$ in a closed form. The analogue of (24) reads

(34) $$\tau_{Bn}\cdot\tau_{Fn}=\sqrt{\det\cosh H}$$

where $\tau_{Bn}=\langle\varphi_B(a_1)\cdots\varphi_B(a_n)\rangle$, $\tau_{Fn}=\langle\varphi_F(a_1)\cdots\varphi_F(a_n)\rangle$, and $G=e^{-2H}$ denote the corresponding solution of II–(18) [2]. The mixed τ-functions

(35) $$\hat{\tau}^{\nu_1,\cdots,\nu_m}_{Bn;\varepsilon_1,\cdots,\varepsilon_m}=\langle\varphi_B(a_1)\cdots\varphi^B_{\varepsilon_1}(a_{\nu_1})\cdots\varphi^B_{\varepsilon_m}(a_{\nu_m})\cdots\varphi_B(a_n)\rangle/\tau_{Bn}$$

where $\varphi^B_{\varepsilon_i}(a_{\nu_i})$ is placed in the ν_i-th position for $i=1,\cdots,m$, are given by (cf. IV–(49)[2])

(36) $$\hat{\tau}^{\nu_1,\cdots,\nu_m}_{Bn;\varepsilon_1,\cdots,\varepsilon_m}=\mathrm{Hafnian}\,(\hat{\tau}^{\nu_j,\nu_k}_{Bn;\varepsilon_j,\varepsilon_k})_{j,k=1,\cdots,m}.$$

Here

(37) $$\hat{\tau}^{\mu\nu}_{Bn;++}=\overline{\hat{\tau}^{\mu\nu}_{Bn;--}}=-f_{\mu\nu}/2m(a_\mu-a_\nu)$$
$$\hat{\tau}^{\mu\nu}_{Bn;+-}=\overline{\hat{\tau}^{\mu\nu}_{Bn;-+}}=-g^{\mu\nu}/2$$

with $\mu\neq\nu$ and $F=(f_{\mu\nu})$, $G^{-1}=e^{2H}=(g^{\mu\nu})$. In particular the 2-point functions are expressible in terms of the solution $\psi(t)=\psi(t\,;0,1/\pi)$

in reference [9] of the equation (23) with $l=0$. Setting $a_1-a_2=te^{i\theta}/2m$ $(t>0)$ we have

(38) $$\tau_{B2}\cdot\tau_{F2}=\cosh(\psi(t)/2)$$

(39) $$\begin{pmatrix}\langle\varphi_+^B(a_1)\varphi_+^B(a_2)\rangle & \langle\varphi_+^B(a_1)\varphi_-^B(a_2)\rangle \\ \langle\varphi_-^B(a_1)\varphi_+^B(a_2)\rangle & \langle\varphi_-^B(a_1)\varphi_-^B(a_2)\rangle\end{pmatrix}=\begin{pmatrix}-ie^{-i\theta}\psi'(t) & -i\sinh\psi(t) \\ i\sinh\psi(t) & ie^{i\theta}\psi'(t)\end{pmatrix}\tau_{B2}/2$$

where $\psi'(t)=\dfrac{d\psi}{dt}$.

References

[1] M. Sato, T., Miwa, and M. Jimbo: Proc. Japan Acad., **53A**, 6–10 (1977).
[2] ——: Ibid., **53A**, 147–152, 153–158, 183–185 (1977).
[3] ——: Ibid., **53A**, 219–224 (1977).
[4] ——: Ibid., **54A**, 1–5 (1978).
[5] ——: Ibid., **54A**, 36–41 (1978).
[6] ——: RIMS preprint, no. **258** (1978).
[7] ——: Publ. RIMS **14**, 223–267 (1977).
[8] ——: RIMS preprint, no. **248** (1978).
[9] B. M. McCoy, C. A. Tracy, and T. T. Wu: J. Math. Phys., **18**, 1058–1092 (1977).