66. Note on Riemann Sum.

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1. Introduction.

B. Jessen has proved the following theorem:

Theorem. *Let $f(x)$ be an integrable function in the interval* (0,1) *and its Riemann sum be*

$$F_n(x) = \frac{1}{n}\sum_{k=1}^{n} f\left(x+\frac{k}{n}\right).$$

Then

$$F_{2^n}(x) \to \int_0^1 f(t)\,dt$$

almost everywhere.

More generally, if n_k divides n_{k+1} for any k, then

$$F_{n_k}(x) \to \int_0^1 f(t)\,dt \tag{1}$$

almost everywhere.

The object of this paper is to prove some related theorems. In Theorem 2 we prove that in Jessen's theorem we can replace the condition that n_k divides n_{k+1} by the Hadamard condition

$$n_{k+1} / n_k > a > 1 \quad (k = 1, 2, \ldots) \tag{2}$$

with an additional condition that

$$(a_n \log n,\ b_n \log n;\ n = 1, 2, \ldots) \tag{3}$$

is a sequence of Fourier coefficients of an integrable function, where $(a_n, b_n; n=1, 2, \ldots)$ being that of $f(x)$. This is derived from Theorem 1 as a special case.

In Theorem 4 we prove that the arithmetic mean of the Riemann sum

$$\frac{1}{n}\sum_{k=1}^{n} F_k(x) \to \int_0^1 f(t)\,dt \tag{4}$$

almost everywhere under the condition of Theorem 2. This is contained in Theorem 3.

Finally, in Theorem 5 and 6 we prove L^p-analogue of Theorem 1 and 3.

In the proof of these theorems we use the method in the paper "S. Izumi and T. Kawata, Notes on Fourier Series (1): Riemann sum," Proc. Imp. Acad., 13(1937), which contains some mistakes so that the stated theorem is not correct.

2. Theorems 1 and 2.

Theorem 1. *Let* (p_n) *be a positive sequence such that* $(1/p_n)$ *is convex and* $1/p_n \to 0$. *Let* $f(x)$ *be an integrable function in* (0,1) *and its Fourier series be*

(5) $$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos 2\pi nx + b_n \sin 2\pi nx).$$

If

(6) $$(a_n p_n,\ b_n p_n;\ n = 1, 2, \ldots\ldots)$$

is a sequence of Fourier coefficients of an integrable function, then (1) *holds for almost all x, where* (n_k) *is taken such as* $\Sigma 1/p_{n_k}$ *converges.*

For example, if $p_n \equiv \log^p n$ $(p > 1)$, then $\Sigma 1/p_{n_k}$ converges when $n_k = [a^k]$ for any $a > 1$. In Jessen's theorem $a = 2$, and in our case n_k does not divides n_{k+1} in general. As a consequence of Theorem 1, we have

Theorem 2. *Let* $f(x)$ *be an ntegrable function in* (0,1) *and its Fourier series be* (5). *If* (3) *is a sequence of Fourier coefficients of an integrable function, then* (1) *holds for* (n_k) *with the Hadamard condition* (2).

We will now prove Theorem 1. By elementary calculation

$$F_k(x) = \frac{1}{k} \sum_{\nu=0}^{k-1} f(x + \nu / k)$$
$$\sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_{nk} \cos 2\pi nk\, x + b_{nk} \sin 2\pi\, nkx).$$

Without loss of generality we can suppose that $a_0 = 0$. Hence we have to prove that $F_{n_k}(x)$ tends to zero almost everywhere.

By the W. H. Young theorem

$$d_k / 2 + \sum_{n=1}^{\infty} \cos 2\pi\, nx / p_{n_k}$$

is a Fourier series of a non-negative integrable function, which we denote by $h_k(x)$, where d_k is taken such that

$$d_k,\quad 1/p_k,\quad 1/p_{2k}$$

is convex and d_k tends to zero as k increases indefinitely.

By the assumption there is an integrable function $g(x)$ such that

$$g(x) \sim \sum_{n=1}^{\infty} (a_n \cos 2\pi\, nx + b_n \sin 2\pi\, nx)\, p_n .$$

Thus we have

$$\int_0^1 h_k(kt)\, g(t-x)\, dt \sim \sum_{n=1}^{\infty} (a_{nk} \cos 2\pi\, nkx + b_{nk} \sin 2\pi\, nkx),$$

and then

$$F_k(x) = \int_0^1 h_k(kt)\, g(t-x)\, dt$$

almost everywhere.

If $g(t)$ is bounded, then there is an M such as $|g(x)| \leqq M$. Hence

$$\left|\int_0^1 h_k(kt)\,g(t-x)\,dt\right| \leqq \int_0^1 h_k(kt)\,g(t-x)\,dt$$

$$\leqq M\int_0^1 h_k(kt)\,dt$$

$$\leqq \frac{M}{k}\int_0^k h_k(t)\,dt = M\int_0^1 h_k(t)\,dt = d_k\,M,$$

which tends to zero as $k\to\infty$.

In the general case, let us put

$$E_m \equiv (t\,;\,|g(t)|>m)\,(n=1,2,\ldots\ldots)$$

whose measure tends to zero as $n\to\infty$. Then

$$\int_0^1\left|\int_{E_m} h_k(k(t+x))\,g(t)\,dt\right|dx \leqq \int_0^1 dx\int_{E_m} h_k(k(t+x))\,|g(t)|\,dt$$

$$=\int_{E_m}|g(t)|\,dt\int_0^1 h_k(k(t+x))\,dx = d_k\int_{E_m}|g(t)|\,dt.$$

Since we can take $d_k \leqq 2/p_k$ for $k\geqq k_0$, Σd_{n_k} converges. Therefore we have

$$\int_0^1\left\{\sum_{k=1}^{\infty}\left|\int_{E_m} h_{n_k}(n_k t)\,g(t-x)\,dt\right|\right\}dx \leqq \left(\sum_{k=1}^{\infty} d_{n_k}\right)\int_{E_m}|g(t)|\,dt$$

which tends to zero as $n\to\infty$. Hence there is a sequence (m_ν) such that

$$\lim_{\nu\to\infty}\sum_{k=1}^{\infty}\left|\int_{E_{m_\nu}} h_{n_k}(n_k t)\,g(t-x)\,dt\right| = 0$$

almost everywhere, and then, for any positive ε, there is a μ such that

$$\left|\int_{E_{m_\mu}} h_{n_k}(n_k t)\,g(t-x)\,dt\right| < \varepsilon$$

almost everywhere for all k.

On the other hand

$$\int_0^1 h_{n_k}(n_k t)\,g(t-x)\,dt = \int_{E_{m_\mu}} h_{n_k}(n_k t)\,g(t-x)\,dt$$

$$+\int_{CE_{m_\mu}} h_{n_k}(n_k t)\,g(t-x)\,dt,$$

where CE denotes the complementary set of E. The second term of the right hand side tends to zero as $k\to\infty$, as was proved. Thus

$$\limsup\left|\int_0^1 h_{n_k}(n_k t)\,g(t-x)\,dt\right| \leqq \varepsilon$$

almost everywhere. Since ε is arbitrary, the theorem is proved.

3. Theorems 3 and 4.

Therem 3. *Let* (p_n) *be a positive sequence such that* $\Sigma 1/np_n$ *converges, and* $(1/p_n)$ *is convex. Let* $f(x)$ *be an integrable function in* (0,1) *and its Fourier series be* (5). *If* (6) *is a sequence of Fourier coefficients of an integrable function, then* (4) *holds almost everywhere.*

For example, if $p_n = \log^p n\,(p > 1)$, then $\Sigma 1/np_n$ converges, and $(1/p_n)$ is convex. Let $f(x)$ be an integradie function in (0,1). Then we get

Theorem 4. *Under the assumtion of Theorem 2,* (4) *holds almost everywhere.*

For the proof of Theorem 3 it is sufficient to prove the convergence of the series

$$\sum_{k=1}^{\infty} F_k(x)/k.$$

Now

$$\Sigma\,|\,F_k(x)\,|\,/\,k = \Sigma\frac{1}{k}\left|\int_0^1 h_k(kt)g(t-x)\,dt\right| \leqq M\Sigma\, d_n/n \leqq M\Sigma 1/np_n < \infty.$$

Thus we get the required.

4. Theorems 5 and 6.

Theorem 5. *Let* (p_n) *be a positive sequence such as* $(1/p_n)$ *is convex and* $1/p_n \to 0$. *Let* $f(x)$ *be an integrable function in* (0,1) *and its Fourier series be* (5) *and* (6) *be a sequence of Fourier coefficients of an integrable function in* L^p $(p>1)$, *then* (1) *holds for almost all* x, *where* (n_k) *is taken such as*

$$\Sigma 1/p_{n_k}^{1+p/q} \qquad (1/p+1/q=1)$$

converges.

For the proof we use the notations in the proof of Theorem 1. By the positiveness of $h_k(t)$ and the Hölder inequality,

$$\left|\int_0^1 h_k(kt)g(t-x)\,dt\right| \leqq \left|\int_0^1 h_k(kt)^{1/q} h_k(kt)^{1/p} g(t-x)\,dt\right| \leqq \left(\int_0^1 h_k(kt)\,dt\right)^{1/q}\left(\int_0^1 h_k(kt)\,|\,g(t-x)\,|^p dt\right)^{1/p},$$

$$\left|\int_0^1 h_k(kt)g(t-x)\,dt\right|^p \leqq d_k^{p/q}\int_0^1 h_k(kt)\,|\,g(t-x)\,|^p dt,$$

$$\sum_{k=1}^{\infty}\int_0^1\left|\int_0^1 h_{n_k}(n_k t)g(t-x)\,dt\right|^p dx = \sum_{k=1}^{\infty} d_{n_k}^{p/q}\int_0^1 dx\int_0^1 h_k(kt)\,|\,g(t-x)\,|^p dt$$

$$\leqq M \sum d_{n_k}^{1+p/q}.$$

Therefore

$$\sum_{k=1}^{\infty} \left| \int_0^1 h_{n_k}(n_k t) g(t-x)\, dt \right|^p$$

is finite almost everywhere. Proceeding as in Theorem 1 we get the required result.

Theorem 6. *In the hypotheses of Theorem 5, if we replace the convergence of* (7) *by that of* $\sum 1/n p_n^p$, *then* (4) *holds.*

[Added in Proof.] We can show that Theorem 2 is best possible in a sense, that is, there exists an integrable function $f(x)$ such that it satisfies the contition of Theorem 2 but its Riemann sum $F_n(x)$ diverges almost everywhere as $n \to \infty$; for example, we may take as $f(x)$ the series $2 + \sum_{n=1}^{\infty} \cos 2\pi nx / n^a$ $(0 < a < 1/2)$.