65. On a Theorem of Banach Space.

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Let E and E' be metric compact spaces, and R and R' be the sets of all real continuous functions on E and E' respectively. If we define addition and multiplication by real numbers by the ordinary method, and the norm by the maximum value of function, then R and R' become Banach spaces.

We owe the following theorem to Banach.(1)

Theorem. *In order that E and E' be homeomorph, it is necessary and sufficent that R and R' are isometric.*

The object of this paper is to give an elementary proof of the theorem.

Since necesity is evident, it remains to prove the sufficiency. Let $V(x) = x'$ be the isometric transformation from R to R'. If we put $U(x) = V(x) - V(\theta)$, then $U(x)$ defines an equivalent transformation(2) from R to R'.

We requires several lemmas.

1°. $U(e) = e'$, where e and e' are units in R and R', that is, the functions which take the value 1 on the whole space E and E' respectively.

Proof. We put $U(e) \equiv e_1' \varepsilon R'$, and prove that $e_1' = e'$, that is $e_1'(s) = 1$ for all $s \varepsilon E'$. From the isometric property $\| e \| = \| e'_1 \| = 1$, that is $\max |e'_1(s)| = 1$. We can assume that $\max e_1'(s) = 1$, for if $\min e_1'(s) = -1$, it suffices to consider $-U(x)$ instead of $U(x)$,

When $e_1'(s) \neq 1$ there exists an $s_0 \varepsilon E'$ such that $e_1'(s_0) = b$, $o < b < 1$. Let us consider a sphere K' in E' with radius σ and center at s_0, and define a continuous function $d'(s) = d'(s_0, a, \sigma)$ such that

$$d'(s) \begin{cases} = a & \text{if } s = s_0 \\ = o & \text{if } s \varepsilon E' - K' \end{cases}$$

and $o \leqq d'(s) < 1$ otherwise. If $a > o$ is sufficiently large, $\sigma > 0$ sufficiently small and $d'(s)$ decreases sufficiently rapidly, as s varies from s_0, then we have

$$\| e' + d' \| = a + b.$$

If we put $d = U^{-1}(d')$, $\max |d(t)| = a$. Then according as $\max d(t) = a$ or $= -a$, we have

(1) Banach, *Théorie des opérations linéaires*, p. 170, Théorème 3.

(2) Banach, loc. cit., p. 181, Théorème 4.

$$\| e_1' + d' \| = \| e + d \| = a + 1,$$

or $$\| e_1' - d' \| = a - b - \varepsilon', \quad \| e - d \| = a + 1,$$

ε' being positive such as $o < b + \varepsilon' < 1$. These both are imposible.

Thus we have the resuired.

2°. $U(ae) = ae'$, where a is an arbitrary real number.

Proof. This is the direct consequence of the homogenuity of $U(x)$. We write the above relation simply by $U(a) = a'$.

3°. $x(t) \geqq 0$ implies $x'(s) \geqq 0$, where $x' = U(x)$.

Proof. $x(t) \geqq 0$ and $\| x \| = a$. If we suppose min $x'(s) = -b < 0$, then we get the following contradiction.

$$a \geqq \| x - a \| = \| x' - a \| = a + b.$$

Thus we get the required.

4°. If a positive function $x(t)$ have its maximum at only one point, then so is for its transform $U(x) = x'(s)$.

Proof. Let us assume that

$$\max. \; x(t) = x(t_0) = a, \quad x'(s_0) = x'(s_1) = a, \quad s_0 \neq s_1.$$

Consider a continuous function d' (s) $= d'(s_0, a, \sigma)$ as in the proof of 1°, such that $s_1 \bar{\varepsilon} K'$ and $d'(s) \leqq x'(s)$ for $s \varepsilon K'$. If we defined y' by

$$y'(s) = x'(s) - d'(s),$$

then

$$y'(s) \geqq 0, \quad y'(s_0) = 0, \quad \max \; y'(s) = a.$$

Further putting $y = U^{-1}(y')$, $d = U^{-1}(d')$, we have

$$x(t) = y(t) + d(t), \quad y(t) \geqq 0, \quad \max \; y(t) = \max \; d(t) = a.$$

If there are different points that make $y(t)$ and $d(t)$ maximum, then $x(t) = y(t) + d(t)$ has two maximums at least, which contradicts the hypothesis. If the maximum point of $y(t)$ coincides with that of $d(t)$, we have max $x(t) = 2a$, which is also impossible.

5°. By 4°, only one maximum point t_0 of $x(t) \geqq 0$ determines the maximum point s_0 of $U(x) = x'(s)$, which gives us a transformation $s_0 = \varphi(t_0)$ from E and E'. This transformation depends only on the positiveness and uniqueness of maximum point of $x(t)$.

Proof. Let $x(t)$ and $y(t)$ be arbitrary non-negative functions taking only one maximum value a at the same point. Let us put

$$U(x) = x', \qquad U(y) = y',$$

$$\max \; x'(s) = x'(s_0), \; \max \; y'(s) = y'(s_1)$$

If $s_0 \neq s_1$, then

$$2a = \| x + y \| = \| x' + y' \| < 2a,$$

which is impossible. Thus we get the required.

6°. If $x(t) \geqq 0$, $x(t_0) = a$. $U(x) = x'$, $\varphi(t_0) = s_0$ where $x(t)$ needs not be maximum at t_0, then $x'(s_0) = a$.

Proof. Let us consider a continuous function $d(t) = d(t_0, a, \sigma)$ such as

$$d(t) \leqq x(t).$$

Putting $x(t) = y(t) + d(t)$, we have

$$y(t) \geqq 0, \qquad y(t_0) = 0.$$

Putting $U(y) = y'$, $U(d) = d'$, we have

$$x'(s) = y'(s) + d'(s), \quad y'(s) \geqq 0.$$

From $\max d(t) = d(t_0) = \max d'(s) = d'(s_0) = a$, we get

$$a = \| \varepsilon y + d \| = \| \varepsilon y' + d' \|$$

provided that ε is an arbitrarily small positive constant. Accordingly we have $\varepsilon y'(s_0) = 0$, that is $y'(s_0) = 0$, and then $x'(s_0) = a$.

7°. $\varphi(t) = s$ is bicontinuous.

Proof. Let $A \subset E$ be an arbitrary closed set, and a continuous function $x(t) \geqq 0$ be such that

$$x(t) \begin{cases} = 1, & \text{if } t \in A, \\ < 1, & \text{if } t \in E-A. \end{cases}$$

Let $U(x) = x'$ and $A' = (s\,;\, x'(s) = 1)$. Obviously A is a closed set in E′ and from 6° $\varphi(A) = A'$, therefore φ is continuous. The continuity of φ^{-1} is clear.

We will now prove the theorem. If we define ordinal multiplication in R and R′, then R and R′ become commutative rings with units e and e′ respectively, and the relations

$$\| xy \| \leqq \| x \| \cdot \| y \|, \quad \| x'y' \| \leqq \| x' \| \cdot \| y' \|$$

hold, Thus R and R′ are normed rings.

6°. Implies $U(xy) = U(x)\,U(y)$. For if $\varphi(t) = s$,

$$(U\,xy))(s) = (xy)(t) = x(t)y(t) = (U(x))(s)\,(U(y))(s).$$

Thus between R and R′ exists an isomorphism as rings. Let $\mathfrak{m}$ and $\mathfrak{m}'$ be the spaces of all maximal ideals of R and R′ respectively. Then the homeomorphisms

$$\mathfrak{m} \sim \mathfrak{m}', \quad \mathfrak{m} \sim E, \quad \mathfrak{m}' \sim E'$$

exist, which implies $E \sim E'$.

(3) Gelfand und G. silov, Über verschiedene Methoden der Einführung der Topologie in die Menge der maximalen Ideale eines normierten Ringes, Recueil math., S. 37 (1940).