34. *Number of Divisor Classes in Algebraic Function Fields.*

By Eizi INABA.

Mathematical Institute, Ochanomizu University, Tokyo.

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For imaginary quadratic fields with discriminant d and with class number h there exists the relation

$$\lim \frac{\log h}{\log \sqrt{|d|}} = 1,$$

if $|d|$ tends to infinity.[1] This result, due to Mr. Siegel, is one of the most interesting in number theory. I will show in this note that the similar relation also holds, when we consider fields of algebraic functions with a finite field of constants.

Let q be the number of elements in a finite field k and g be the genus of a function field K with k as field of constants. If z is an element in K, which is not contained in k, then K becomes a finite extention of $k(z)$. For any integral divisor A of K, whose degree is λ, put $N(A) = q^{\lambda}$. s being a complex variable, the series

$$\zeta(s) = \sum \frac{1}{N(A)^s}$$

is called the ζ-function of K, where the summation extends over all integral divisor A of K. The ζ-function $\zeta_0(s)$ of the field $k(z)$ becomes

$$(1) \qquad \zeta_0(s) = \sum_{n=0}^{\infty} \frac{q^{n+1}-1}{q-1} \frac{1}{q^{ns}} = \frac{1}{\left(1-\frac{q}{q^s}\right)\left(1-\frac{1}{q^s}\right)}$$

It is well known that the quotient $L(s)$ of $\zeta(s)$ divided by $\zeta_0(s)$ is a polynomial with respect to $\frac{1}{q^s}$ and

$$L(s) = 1 + \frac{N_1-g+1}{q^s} + \dots + \frac{q^g}{q^{2gs}},$$

where N_1 is the number of prime divisors of K with degree 1.[2] According to the so-called Riemann's conjecture for function fields proved by Weil and Igusa, the real parts of all zero-points of $L(s)$

1) C. L. Siegel: Über die Klassenzahl quadratischer Zahlkörper, Acta Arith. 1 (1935).

2) H. Hasse: Über die Kongruenzzetafunktionen, Sitzungsberichte der Preussischen Akademie der Wissenschaften, 1934.

are equal to $\frac{1}{2}$.[3] Therefore, putting

$$(2) \qquad L(s) = \prod_{i=1}^{2g}\left(1-\frac{\omega_i}{q^s}\right)$$

the absolute values of all ω_i are $\sqrt{q}$. Let N_λ, n_λ be respectively the number of prime divisors of K, $k(z)$ with degree λ. Then we can modify Reichardt's estimation for N_λ and n_λ as follows[4]

$$(3) \qquad \left|n_\lambda - \frac{q^\lambda}{\lambda}\right| < 2q^{\frac{\lambda}{2}}$$

$$(4) \qquad \left|N_\lambda - n_\lambda\right| \leqq 4gq^{\frac{\lambda}{2}}$$

If we denote with h the number of divisor classes of K with degree zero, then the number of divisor classes with an arbitrarily given degree is also h. Since the number of integral divisors of K with degree $2g$ equals to

$$h\,\frac{q^{g+1}-1}{q-1}$$

by Riemann-Roch's theorem, we have from (3), (4)

$$h\,\frac{q^{g+1}-1}{q-1} > N_{2g} > \frac{q^{2g}}{2g} - (4g+2)\,q^g\,.$$

If k is fixed and g tends to infinity, we have

$$(5) \qquad \underline{\lim}\,\frac{\log h}{g\log q} \geqq 1\,,$$

whence we can assert that h tends to infinity, if g does so. This is essentially the extension for function fields of the result proved by Heilbronn and Siegel: the class number of imaginary quadratic field tends to infinity, if the absolute value of its discriminant does so.[5]

In order to obtain a similar result for K as Siegel's, we must put the following condition for K. *It is possible to choose an element z in K, such that the degree $(K : k(z))$ of K over $k(z)$ does not surpass*

3) A. Weil: Sur les fonctions algébriques à corps de constantes fini, Comptes Rendus, Vol. 210 (1940).

A. Weil: On the Riemann hypothesis in the function fields, Proc. of the Nat. Acad. of Sciences, Vol. 27 (1941).

J. Igusa: On the theory of algebraic correspondences and its application to the Riemann hypothesis in function-fields, Journal of the Math. Soc. of Japan, Vol. I, No. 2 (1949).

4) H. Reichardt: Der Primdivisorsatz für algebraische Funktionenkörper über einem endlichen Konstantenkörper, Math. Zeitschr. Bd. 40 (1936).

5) H. Heilbronn: On the class-number in imaginary quadratic fields, Quarterly Journal, Oxford, Ser. 5 (1934).

a given positive integer $m>1$. *Under this condition it holds the relation*

$$\lim_{g\to\infty}\frac{\log h}{g\log q}=1. \tag{6}$$

The proof runs as follows. We denote with $d(A)$ and $n(A)$ respectively the dimension and the degree of an integral divisor A of K. Riemann-Roch's theorem yields

$$d(A)\geqq n(A)-g+1.$$

Therefore, if $s>1$, we obtain

$$\begin{aligned}\zeta(s)&\geqq h\sum_{\lambda=g}^{\infty}\frac{q^{\lambda-g+1}-1}{q-1}\cdot\frac{1}{q^{\lambda s}} \\ &=\frac{h}{q^{gs}\left(1-\frac{q}{q^s}\right)\left(1-\frac{1}{q^s}\right)}=\frac{h}{q^{gs}}\zeta_0(s).\end{aligned} \tag{7}$$

Now $\zeta(s)$ can be represented in the following manner

$$\zeta(s)=\Pi\frac{1}{1-\frac{1}{N(P)^s}},$$

where P runs over all prime divisors of K and $s>1$. The similar representation is also possible for $\zeta_0(s)$. For a prime divisor P of K with relative degree t, which is generated from a prime divisor P_0 of $k(z)$, we have

$$1-\frac{1}{N(P)^s}\geqq\left(1-\frac{1}{N(P_0)^s}\right)^t.$$

Therefore, if $P_1, P_2, \ldots, P_r$ are all prime divisors of K generated from P_0, it holds

$$\prod_{i=1}^{r}\left(1-\frac{1}{N(P_i)^s}\right)\geqq\left(1-\frac{1}{N(P_0)^s}\right)^m,$$

whence $\zeta_0(s)^m\geqq\zeta(s)$ follows. This together with (7) yields

$$\zeta_0(s)^{m-1}\geqq\frac{h}{q^{gs}}.$$

As m and q are fixed, it follows then

$$s\geqq\overline{\lim}\frac{\log h}{g\log q},$$

when g tends to infinity. Since the value of s can be taken arbitrarily near to 1, we have finally

$$1\geqq\overline{\lim}\frac{\log h}{g\log q}.$$

This together with (5) yields the required relation (6).

Remark 1. The above obtained asymptotic relation of h with g has really its meaning. If we consider function fields generated by adjoining to $k(z)$ the m-th roots of polynomials in $k[z]$, then we ascertain that there exist function fields with any large genus satisfying the above mentioned condition. That h is not uniquely determined, when the value of q, g, m are fixed, may be conceivable by the following examples. k being a finite field with $q=3$, consider the following function fields

$$k\,(z,\sqrt{z^3+z+2})\,,\qquad k\,(z,\sqrt{(z^2+1)(z^2+z+2)}\,)\,,$$
$$k\,(z,\sqrt{z\,(z+1)(z^2+1)}\,)\,.$$

Then we have $m=2$, $g=1$ for each fields, but the value of h is respectively 3, 4, 6.

Remark 2. It is difficult to see if the relation (6) subsists, when the condition $(K:k(z))\leqq m$ is removed. However we can determine the scope of the values of h referred to g and q as follows. Since the residues at $s=0$ of the functions $\zeta(s)$, $\zeta_0(s)$ are respectively

$$-\frac{h}{(q-1)\log q}\quad\text{and}\quad-\frac{1}{(q-1)\log q}$$

we have

$$h=L(0)=\prod_{i=1}^{2g}(1-\omega_i)$$

The equalities $|\omega_i|=\sqrt{q}$ give rise to

$$(\sqrt{q}+1)^{2g}\geqq h\geqq(\sqrt{q}-1)^{2g}$$

Remark 3. This remark is due to Mr. Iwasawa, who kindly read through my manuscript. K being separable over $k(z)$, let D be the different (divisor) of K referred to $k(z)$, Riemann's formula yields

$$2g-2=n(D)-2\,(K:k(z))\,.$$

So we can modify (6) as follows

$$\lim\frac{\log h}{\log\sqrt{N(D)}}=1$$

in accordance with Siegel's result.