38. The Two-sided Representations of an Operator Algebra.

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The object of the present note is to investigate the relation between the two-sided representations and the traces of a uniformly closed operator algebra on a Hilbert space (i.e. C*-algebra in the tirminology of I.E. Segal [7]). Our investigation is closedly connected with the recent works of R. Godement [2], I. E. Segal [9] and J. Dixmier [1].

1. We suppose that R is a C*-algebra *having the identity* 1 (with elements x, y, z, etc.) and ω, σ, τ etc. are the *states* of R (i. e. line functionals on R, considering as a Banach space, with $\omega(xx^*) \geqq 0$ for all x and $\omega(1) = 1$). A *trace* of R is a state which satisfies moreover $\tau(xy) = \tau(yx)$ for any pair x and y. If for any x there exists a trace τ such that $\tau(xx^*) > 0$, then we say that R has *sufficiently many traces* (or shortly is of the *trace type*). The *state space* of all states is a convex and weakly* closed subset in the unit sphere of the conjugate space of R. Also it is easy to see that the set T (the *trace space*) of all traces forms a convex and weakly* closed subset in the state space. Whence by the well-known theorem of Tychonoff, they are compact in the (bounded) weak* topology of the conjugate space. It is an easy consequence of the theorem due to M. Krein and D. Milman [3] that *a* C*-*algebra has sufficiently many traces if and only if it has sufficiently many characters* where we mean by a *character* an extreme point of the trace space.

Concerning the notion of the trace type, the following observation may have some interest. If the "Poisson bracket" $[x, y] = i(xy-yx)$ of any pair of hermitean clements x and y is not strictly positive definite then we will call that the algebra is of *semi-trace type*. This terminology is justified by the following

THEOREM 1. *A C*-algebra is of semi-trace type if and only if it has at least one trace.*

Since the proof of this theorem can be done in somewhat similar manner to that of our preceeding paper [5], we may omit the detail.

2. Let now H be a Hilbert space with elements ξ, η, ζ, etc. In this space we now introduce the following

DEFINITION. An *involution* j is a conjugate linear transformation of period two onto itself with $(\xi j, \eta j) = (\eta, \xi)$. For a C^*-algebra R, a mapping $x^\flat$ form R into the operator algebra on H is called a *dual representation* of R provided that $(xy)^\flat = y^\flat x^\flat$ holds instead of the usual representation of R in the sense of I. E. Segal [7]. A pair of a representation $x^\sharp$ and a dual representation $x^\flat$ of R on same H is called a *two-sided representation* of R provided that there exists an involution j with (1) $x^\sharp y^\flat = y^\flat x^\sharp$ and (2) $x^{\sharp *} = jx^\flat j$.

Clearly this is a generalization of the definition of R. Godement [2] which he defines for the pair of unitary representations of a group.

3. In this and the next sections, we will consider the relation between the irreduciblity of the two-sided representations and the traces of a C^*-algebra following the line of I. E. Segal [7] with some modifications.

Suppose that a C^*-algebra R has a trace τ. Since τ is also a state, by the method of Segal we can construct a representation $x^\sharp$ of the algebra as follows: Let I be the set of all elements x such as $\tau(xx^*) = 0$. Then I is a (two-sided) ideal (whence R/I is a C^*-algebra by a theorem of I. E. Segal [8]). Introducing an inner product $(x^\theta, y^\theta) = \tau(xy^*)$ (where x^θ is the residue class containing x), R/I becomes an incomplete Hilbert space. Hence there exists a Hilbert space H which is the metrical hull of R/I. In H we can define $x^\sharp$ as a continuation of the operator $y^\theta x^\sharp = (yx)^\theta$ on R/I. This is the required representation of the algebra such that $1^\theta R^\sharp = R/I$ is dense in H (i.e., *normal* in the sence of Segal) and $\tau(x) = (1^\theta x^\sharp, 1^\theta)$.

On the other hand, R is also representable in the left considering R/I by way of a modul having R as left operator-domain, that is, x' can be defined by $x'y^\theta = (xy)^\theta$ (the left representation). Let now define $x^\flat$ as $y^\theta x^\flat = x'y^\theta$. Then clearly $x^\flat$ is a dual representation of the algebra in the above sense. Moreover, if we introduce the involution j as $x^\theta j = x^{*\theta}$ in R/I (and its extention), then the following identities

$$z^\theta x^\sharp y^\flat = (zx)^\theta y^\flat = (yzx)^\theta = (yz)^\theta x^\sharp = z^\theta y^\flat x^\sharp,$$
$$z^\theta jx^\flat j = (xz^*)^\theta j = (zx^*)^{*\theta} j = (zx^*)^\theta = z^\theta x^{\sharp *}$$

imply that the pair $x^\sharp$ and $x^\flat$ is a two-sided representation.

These proved the following

THEOREM 2. *If a C^*-algebra has a trace, then there exists a normal two-sided representation whose normalizing function coincides with the given trace. This representation is determined within the isomorphisms.*

This theorem may justify the term "two-sided representation".

4. We now turn to investigate the relation between the irreducibility and the characters of the algebra. As usually we define the *irreducibility* of a two-sided representation provided that no proper subspace of H exists which is invariant under $R^{\sharp}$, $R^{\flat}$ and j.

THEOREM 3. *The two-sided representation generated by a character is irreducible and conversely.*

PROOF. Suppose that a reducible two-sided representation is generated by a character χ. Then there exists a projection e which commutes with both $R^{\sharp}$ and $R^{\flat}$ and j. Put $\tau'(x) = (1^{\theta}ex^{\sharp}, 1^{\theta}e)$ and $\tau''(x) = (1^{\theta}(1-e)x^{\sharp}, 1^{\theta}(1-e))$. They are scalar multiples of states.

$$\tau'(xy) = (1^{\theta}ex^{\sharp}y^{\sharp}, 1^{\theta}e) = (x^{\theta}e, y^{*\theta}) = (x^{\theta}e, y^{\theta}j) = (y^{\theta}e, x^{\theta}j) = \tau'(yx)$$

implies that they are also multiples of traces. Since $\chi = \tau' + \tau''$ and x is the extreme point of T, we have $\chi = \alpha\tau'$ for some real α. Now the remainder of the proof runs on the same line as Segal [7].

The proof of the converse is also similar to that of Segal. Only troublesome effect occures by the existence of the involution. But this is excluded by the identity:

$$(x^{\theta}jaj, y^{\theta}) = (y^{\theta}j, x^{\theta}ja) = \sigma(y^{*}x^{**}) = \sigma(y^{*}x) = \sigma(xy^{*}) = (x^{\theta}a, y^{\theta})$$

where σ is a trace and a is the non-negative definite operator of H defined by $(x^{\theta}a, y^{\theta}) = \sigma(xy^{*})$.

Now, the following is an easy consequence of the above theorem and the well-known theorem of J. von Neumann which states that a W^{*}-algebra (weakly closed algebra) is generated by its projections: *None of operators other than scalars commutes with the representations and the involution when an irreducible two-sided representation is generated by a trace.*

5, The following theorems are direct consequences of the preceeding sections:

THEOREM 4. *A simple C^{*}-algebra has at most one trace.*

PROOF. Since the algebra R is simple and has a trace, it has a character χ and the set of all x with $\chi(xx^{*}) = 0$ vanishes since otherwise it becomes a proper ideal. Therefore, by putting $(x, y) = \chi(xy^{*})$, R becomes a space having an inner product. Suppose that τ is a trace of R and H is the metrical hull of R, then the Segal's technique implies that there exists a semi-positive definite operator a of H with $\tau(xy^{*}) = (xa, y)$. a commutes with $R^{\sharp}$, $R^{\flat}$ and j as in the above. Hence a is a multiple of the identity by Theorem 3 since χ is a character, that is, τ coincides with χ by the definition. This proves the theorem.

THEOREM 5. *If χ is a character of a C^*-algebra, then the set of all x such as $\chi(xx^*)=0$ is a maximal ideal.*

PROOF. It is sufficient to show that the algebra is simple if the set of all x with $\chi(xx^*)=0$ vanishes. Suppose that J is a non-trivial ideal of R. Let now the space H and the inner product as in the above. Since J is non-trivial, by a theorem due to I. E. Segal [8] there exists a state ω which vanishes on J. By a technique of Segal which we have already frequently used, there exists a semi-positive definite operator a on H satisfying $\omega(xy^*)=(xa, y)$. Using the Schwarz inequality we have $(xa, y)=\omega(xy^*)=0$ for any x in J and y in R. Hence J lies in the null space of the operator a. Since the null space of a does not cover the full space, the closure of J (with respect to the metric of H) is a proper subspace of H. Since the two-sided ideal J is invariant under right (left) multiplication and the involution j by a theorem of I. E. Segal [8], its closure is also invariant. On the other hand, χ is a character of the algebra, whence H has no proper invariant subspace by Theorem 3. This contradiction proves the theorem.

COROLLARY. *A character of a C^*-algebra is a non-trivial homomorphism of the center to the complex number field.*

PROOF. Since χ vanishes on a maximal ideal M of the algebra, then the intersection of M and the center is also a maximal ideal in the center. Obviously, χ is not trivial on the center. Therefore, it determines a non-trivial homomorphism of the center to the complex number field.

THEOREM 6. *A C^*-algebra of trace type is strongly semi-simple in the sense of I. Kaplansky* [3].

PROOF. Since the algebra has sufficiently many characters, there exists a maximal ideals which excludes the given element. Hence the intersection of all maximal ideals vanishes.

Theorem 6 implies that the set of all maximal ideals defined by characters of a C^*-algebra of trace type is dense in the spectrum (in the sense of I. E. Segal [6]).

It is also to be noted that a W^*-algebra of finite type in the sence of J. Dixmier [1] is of trace type as already pointed out by R. Godement [2].

References.

1) J. Dixmier: *Mesure de Haar et trace d'un opérateur*, C.R. 228 (1949), 152-154.

2) R, Godement: *Sur la théorie des caractères*, I et II, ibid., 229 (1949), 976-979, 1050-1051.

3) I. Kaplansky: *Normed algebras*, Duke M.J., 16 (1948), 399-418.

4) S. Krein and D. Milman. *On extreme points of regularly convex sets,* Studia Math., 9 (1940), 131–137.

5) M. Nakamura: *Notes on Banach space* (XII), to appear in Tôhoku M.J.

6) I. E. Segal: *The group algebra of a locally compact group,* Trans. Amer. Math. Soc., 61 (1947), 69–109.

7) I. E. Segal: *Irreducible representation of operator algebras,* Bull. Amer. Math. Soc., 53 (1947), 73–88.

8) I. E. Segal: *Two-sided ideals in operator algebras,* Ann. of Math., 50 (1949), 856–865.

9) I. E. Segal: *The two-sided regular representation of a unimodular locally compact group,* ibid., 51 (1950).