256. Pseudo-differential Operators in the Theory of Hyperfunctions

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In this note we first give the definition of the sheaf of pseudo-differential operators, the notion of which is originally due to Sato. The sheaf is defined using Sato's theory of the sheaf $\mathcal{C}$. For the definition and the main properties of the sheaf $\mathcal{C}$ we refer the reader to Sato [2].

Secondly we develop the theory of pseudo-differential operators of finite type and discuss some applications to the theory of linear partial differential operators.

This note is a summary of our forthcoming paper in which the details will be given.

1. Let f be a real analytic mapping from an n-dimensional real analytic manifold N to an m-dimensional real analytic manifold M. The mapping f defines a natural homomorphism of vector bundles: $N \underset{M}{\times} T^*M \to T^*N$, where T^*M and T^*N denote the cotangent bundle of M and that of N, respectively, and $N \underset{M}{\times} T^*M$ denotes the fibre product of T^*M and N over M. We denote its kernel by T^*_NM, which is not a vector bundle in general, and define S^*_NM to be $(T^*_NM - N)/\boldsymbol{R}^+$. ($\boldsymbol{R}^+$ denotes the multiplicative group of positive numbers.) S^*_NM is a closed subset of $N \underset{M}{\times} S^*M$, whose fibres over N are spheres. The $f^* : N \underset{M}{\times} T^*M \to T^*N$ induces a projection $\rho : N \underset{M}{\times} S^*M - S^*_NM \to S^*N$. We denote by $\tilde{\omega}$ the natural projection $N \underset{M}{\times} S^*M - S^*_NM - S^*M$.

Let $\mathcal{A}_M$ be the sheaf of real analytic functions and let v_M be the sheaf of densities with real analytic coefficients, which becomes an invertible $\mathcal{A}_M$-Module. By means of the theory of $\mathcal{C}$, we can define the following two fundamental homomorphisms, corresponding to the substitution and integration along fibre.

$$f^* : \rho_! \tilde{\omega}^{-1} \mathcal{C}_M \to \mathcal{C}_N,$$
$$f_* : \tilde{\omega}_! \rho^{-1} (\mathcal{C}_N \underset{\mathcal{A}_N}{\otimes} v_N) \to \mathcal{C}_M \underset{\mathcal{A}_M}{\otimes} v_M.$$

Remark. If $f : X \to Y$ is a continuous map and $\mathcal{F}$ is a sheaf on X, $f_!(\mathcal{F})$ is a direct image with proper support, that is, $\Gamma(U, f_!(\mathcal{F}))$

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$=\{s \in \Gamma(f^{-1}U; \mathcal{F}); \operatorname{supp}(s)$ is proper over $U\}$, for every open subset U of Y.

We identify N with the submanifold of $N \times M$ by the diagonal mapping. The projection $T^*(N\times M) \to N \times T^*M$ induces an isomorphism $S^*_N(N\times M) \to N \underset{M}{\times} S^*M$, by which we identify $S^*_N(N\times M)$ and $N \underset{M}{\times} S^*M$. In general, if $f: X \to Y$ is a fibre space with sphere as fibre, we denote by a the antipodal mapping of X, which maps $x \in X$ to its antipodal point x^a. For any sheaf $\mathcal{F}$ over X, we denote by $\mathcal{F}^a$ the direct image of $\mathcal{F}$ by a.

Definition 1. We define the sheaf $\mathcal{P}_f$ of pseudo-differential operators over f by $\mathcal{P}_f = \Gamma_{S^*_N(N\times M)}(\mathcal{C}_{N\times M} \underset{\mathcal{A}_M}{\otimes} v_M)^a$. Since the support of $\mathcal{P}_f$ is in $S^*_N(N\times M)$, we consider $\mathcal{P}_f$ as a sheaf on $N \underset{M}{\times} S^*M$.

If $M=N$ and f is the identity map, $\mathcal{P}_f$ is denoted by $\mathcal{P}_M$ or by $\mathcal{P}$ and a section of $\mathcal{P}_M$ is said to be pseudo-differential operator on S^*M. There are two fundamental bilinear homomorphisms

$$\rho_!(\mathcal{P}_f|_{N \underset{M}{\times} S^*M - S^*_N M} \times \tilde{\omega}^{-1}\mathcal{C}_M) \to \mathcal{C}_N$$
$$\tilde{\omega}_!(\rho^{-1}\mathcal{C}_N \otimes v_N \times \mathcal{P}^a_f|_{N \underset{M}{\times} S^*M - S^*_N M}) \to \mathcal{C}_M \otimes v_M.$$

Roughly speaking, the first bilinear homomorphism is defined by $(K(y,x)dx, \varphi(x)) \mapsto \int K(y,x)\varphi(x)dx$ and, the latter by $(\psi(y)dy, K(y,x)dx) \mapsto \left[\int \psi(y)K(y,x)dx\right]dy$.

If $f: N \to M$ and $g: M \to L$ are two real analytic mappings, we can define the composition homomorphism of pseudo-differential operators over g and pseudo-differential operators over f. Let $p_1: N \underset{L}{\times} S^*L - N \underset{M}{\times} S^*_M L \to N \underset{M}{\times} S^*M$ and $p_2: N \underset{L}{\times} S^*L - N \underset{M}{\times} S^*_M L \to M \underset{L}{\times} S^*L$ be the projections deduced from $\tilde{\omega}: M \underset{L}{\times} S^*L - S^*_M L \to S^*M$ and $f: N \to M$. Then the composition of pseudo-differential operators is described by the following bilinear homomorphism: $p_1^{-1}\mathcal{P}_f \times p_2^{-1}\mathcal{P}_g \to \mathcal{P}_{g\circ f}$.

Now, consider the case where $N=M$ and f is the identity. In this case $\mathcal{P}$ has a structure of sheaf of rings by the above composition homomorphism, which reduces to a sheaf bilinear homomorphism $\mathcal{P} \times \mathcal{P} \to \mathcal{P}$ since $S^*_N M$ is empty. By the operations $\mathcal{P} \times \mathcal{C} \to \mathcal{C}$ and $(\mathcal{C}\otimes v) \times \mathcal{P}^a \to \mathcal{C}\otimes v$, $\mathcal{C}$ may be considered as a left $\mathcal{P}$-Module, and $\mathcal{C}\otimes v$ as a right $\mathcal{P}$-Module.

2. We next study a special class of pseudo-differential operators which is easy to manipulate. In the following we fix some local coordinate system and identify M with a domain Ω in $\boldsymbol{R}^n$ and S^*M with $\Omega \times S^{n-1}$. We denote a point in S^*M by (x, η), where η is a homogeneous coordinate of S^{n-1}.

Definition 2 (Pseudo-differential operators of finite type). We call a pseudo-differential operator P to be of finite type when its kernel function $K(x, x')$ is represented near (x_0, η_0) as $\int \sum_{j \geqq -m} p_j(x, \eta)\Phi_j(\langle x-x', \eta\rangle + i0)\omega(\eta)$ where $p_j(x, \eta)$ and $\Phi_j(\tau)$ satisfy the following conditions (i) $\sim$(iii), and j runs over the integers. $\omega(\eta)$ is the volume element $\sum_{j=1}^{n}(-1)^{j-1}\eta_j d\eta_1 \wedge \cdots \wedge d\eta_{j-1} \wedge d\eta_{j+1} \wedge \cdots \wedge d\eta_n$:

(i)
$$\Phi_j(\tau) = \begin{cases} \dfrac{(-1)^j(n-j-1)!}{(-2\pi\sqrt{-1})^n} \cdot \dfrac{1}{\tau^{n-j}} & (j<n) \\ \dfrac{-1}{(2\pi\sqrt{-1})^n(j-n)!}\tau^{j-n}\log\tau & (j \geqq n) \end{cases}$$

(ii) $p_j(x, \eta)$ is real analytic near (x_0, η_0) and positively homogeneous of degree $(-j)$ with respect to η, that is, $p_j(x, c\eta) = c^{-j}p_j(x, \eta)$ $(c>0)$.

(iii) There exist some complex neighbourhood V of (x_0, η_0) and some constant c such that the following estimate (*) holds;

(*) $\sup_{(z,\zeta)\in V} |p_j(z, \zeta)| \leqq j!c^j$ for sufficiently large j.

Definition 3. Let P be a pseudo-differential operator of finite type whose kernel function has the above form. Then we define the order of P by $(-1)\times\min\{j \mid p_j(x, \eta) \not\equiv 0\}$. If P is of order m, then we define the principal symbol of P to be $p_{-m}(x, \eta)$.

Afterwards we sometimes write P as $\sum_{j \geqq -m} p_j(x, D_x)$.

Proposition 4. *Let the kernel function of P be given by*

$$\int \sum_{k \geqq -m} a_k(x, x', \eta)\Phi_k(\langle x-x', \eta\rangle + i0)\omega(\eta),$$

where $a_k(x, x', \eta)$ is real analytic near (x_0, x_0, η_0), is positively homogeneous of degree $(-k)$ with respect to η and satisfies the estimate $\sup_{(z,z',\zeta)\in V} |a_k(z, z', \zeta)| \leqq k!c^k$ $(k \gg 0)$, for some constant c and for some complex neighbourhood V of (x_0, x_0, η_0). Then the kernel function of P has the form stated in Definition 2, that is $\int \sum p_j(x, \eta)\Phi_j(\langle x-x', \eta\rangle + i0)\omega(\eta)$, where

$$p_j(x, \eta) = \sum_{j=k+|\alpha|} \frac{1}{\alpha!} D_{x'}^{\alpha} D_{\eta}^{\alpha} a_k(x, x', \eta)|_{x'=x}.$$

This proposition is proved using the Taylor expansion of a_k with respect to $(x-x')$ and Cauchy's integral formula.

Corollary 1. *The property that a pseudo-differential operator P is of finite type is independent of the choice of the local coordinate system.*

Corollary 2. *The principal symbol of P is independent of the choice of the local coordinate system, as a homogeneous function on S^*M.*

Theorem 5. *Let P and Q be pseudo-differential operators of finite type expressed in the form $\sum p_j(x, D_x)$ and $\sum q_k(x, D_x)$, respectively. Then the composed operator $R=PQ$ is also of finite type and if we write $R=\sum r_l(x, D_x)$, then*

$$r_l(x, \eta)=\sum_{l=j+k+|\alpha|} \frac{1}{\alpha!} D_\eta^\alpha p_j(x, \eta) D_x^\alpha q_k(x, \eta).$$

To prove this theorem we use the following formula which holds as an identity of sections of the sheaf $\mathcal{C}$;

$$\int(\langle x, \xi\rangle+i0)^\lambda(t-\langle x, \eta\rangle+i0)^\mu dx = \frac{(-2\pi i)^n \Gamma(-\lambda-\mu-n)}{\Gamma(-\lambda)\Gamma(-\mu)}(t+i0)^{\lambda+\mu+n}\delta_\lambda(\xi:\eta),$$

where $\delta_\lambda(\xi:\eta)$ is a δ-function concentrated on $\xi=c\eta(c>0)$, homogeneous of degree λ in ξ and of degree $(-n-\lambda)$ in η.

Theorem 6. *Let P be a pseudo-differential operator of finite type and $p_{-m}(x, \eta)$ be its principal symbol. Then* $\operatorname{supp}\operatorname{Ker}_{\mathcal{C}} P$ *and* $\operatorname{supp}\operatorname{Coker}_{\mathcal{C}} P$ *are both contained in $\{(x, \eta) \in S^*M \mid p_{-m}(x, \eta)=0\}$, where* $\operatorname{Ker}_{\mathcal{C}} P$ *is by definition the solution sheaf of the pseudo-differential operator P which operates on the sheaf $\mathcal{C}$, and* $\operatorname{Coker}_{\mathcal{C}} P$ *denotes the cokernel of P.*

This can be proved by the majorant method using the calculation of Theorem 5.

Remark. This theorem was first found by Sato [1] for partial differential operators and is called Sato's fundamental theorem.

Theorem 7. *Let P and Q be partial differential operators of order m with simple characteristics and $P_{-m}(x, \eta)=Q_{-m}(x, \eta)$ is a polynomial of (x, η) (for some local coordinate system). Then we can find locally some invertible pseudo-differential operators of finite type R and S such that $RQ=PS$.*

Remark. The assumption that P and Q are partial differential operators and that $P_{-m}(x, \eta)=Q_{-m}(x, \eta)$ is a polynomial is very unsatisfactory. We hope that the assumption can be relaxed at least to the condition that P and Q are pseudo-differential operators of finite type of order m with simple characteristics and $P_{-m}=Q_{-m}$. In fact, we can find R and S formally in that case but we have not yet proved the estimate $(*)$ in Definition 2.

Sketch of the proof of Theorem 7.

We can assume that $Q(x, \eta)=Q_{-m}(x, \eta)$ without loss of generality. If $P_{-m}(x, \eta)\neq 0$ then Theorem 6 proves this theorem. So we can assume $\operatorname{grad}_\eta P_{-m}\neq 0$ by the assumption of simple characteristics. (In fact, it is sufficient to assume $\operatorname{grad}_{(\eta, x)} P_{-m}(x, \eta)=(\operatorname{grad}_\eta P_{-m}(x, \eta), \operatorname{grad}_x P_{-m}(x, \eta))$ is not parallel to $(0, \eta)$ if $P_{-m}(x, \eta)=0$ instead of the above assumption.) Assuming $R=S=\sum_{j\geqq 0} r_j(x, D_x)$ we obtain successively the first order partial differential equations of the following type:

$$\langle \mathrm{grad}_\eta P_{-m}(x,\eta), \mathrm{grad}_x r_j\rangle - \langle \mathrm{grad}_\eta r_j, \mathrm{grad}_x P_{-m}\rangle + P_{-m+1}(x,\eta) r_j(x,\eta) = \rho_j(x,\eta),$$

where $\rho_0=0$ and $\rho_j(x,\eta)(j\geqq 1)$ is determined by $\{r_k\}_{k=0,\cdots,j-1}$. By the assumption on $\mathrm{grad}_\eta P_{-m}$ we can find some non-characteristic surface $T=\{t(x,\eta)=0\}$ and give the initial condition of r_j on T by 1 for $j=0$ and by 0 for $j\geqq 1$. Using Euler's identity for homogeneous functions we can find $\{r_j(x,\eta)\}$ satisfying the conditions of Definition 2.

References

[1] M. Sato: Hyperfunctions and partial differential equations. Proc. Int. Conf. on Functional Analysis and Related Topics, Univ. of Tokyo Press, pp. 91–94 (1969).

[2] ——: On the structure of hyperfunctions. Sûgaku no Ayumi, **15**, 9–72 (1970) (Notes by M. Kashiwara, in Japanese).