128. *Integral Transforms and Self-dual Topological Rings*

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(Comm. by Z. SUETUNA, M.J.A., Nov. 12, 1960)

It is well known that a generalization of the Poisson summation formula holds on some types of topological groups [1, 3]. In this paper we shall show that if the Poisson summation formula holds in some sense on a locally compact topological ring then the ring is self-dual as an additive group (Proposition 2). In this paper we shall use the following notations:

R is a locally compact ring with a neutral element 1,

R^+ is the additive group composed of all elements of R,

$\hat{R}$ is the dual group of R^+,

μ is a Haar-measure on R^+.

To any measurable functions $f(x)$, $g(x)$, $T(x)$ defined on R $f*g$ is the convolution of f and g on R^+,

$\mathrm{Car}(f)$ is the carrier of f and

$$Tf(x)=\int_R T(xy)f(y)d\mu(y).$$

Finally $\mathfrak{D}^0$ is the set of all continuous functions with compact carrier defined on R.

§1. **Proposition 1.** *Let $T(x)$ be a bounded continuous function on G but be not constant* 0. *If*

$$T(f*g)=Tf\cdot Tg \qquad \textit{for all } f, g\in\mathfrak{D}^0, \tag{1}$$

then $T\in\hat{R}$.

Proof. Let us denote $f_u(x)=f(x+u)$ and $P_f(-u)=\dfrac{Tf_u(1)}{Tf(1)}$. (Naturally P_f is defined to f such that $Tf(1)\neq 0$.) By the hypothesis and the definition of the convolution we have

$$Tf_u\cdot Tg=T(f_u*g)=T(f*g_u)=Tf\cdot Tg_u,$$

and then $P_f(-u)=P_g(-u)$. Therefore we shall denote simply $P(-u)$.

Concerning the function $P(u)$ we get

$$P(u+v)=P(u)P(v), \tag{2}$$

for $P(-u-v)=\dfrac{T(f*f)_{u+v}(1)}{T(f*f)(1)}=\dfrac{T(f_u*f_v)(1)}{T(f*f)(1)}=\dfrac{Tf_u(1)\cdot Tf_v(1)}{Tf(1)\cdot Tf(1)}=P(-u)P(-v).$

For any positive number ε and any $f\in\mathfrak{D}^0$ there exists an open set of R such that

$$|Tf_u(1)-Tf(1)|\leqq\int_R|T(x)||f(x+u)-f(x)|\,d\mu(x)<\varepsilon$$

if u belongs to this set, because $f(x)$ has the compact carrier. From this and (2) we can conclude that $P(x)$ is a continuous function.

By the definitions we have

$$Tf_u(1)=P(-u)Tf(1)=\int_R P(-u)T(x)f(x)d\mu(x)$$

and

$$Tf_u(1)=\int_R T(x)f(x+u)d\mu(x)$$
$$=\int_R T(x-u)f(x)d\mu(x)$$

for all $f\in\mathfrak{D}^0$. So we get

$$P(-u)T(x)=T(x-u).$$

In particular

(3) $$T(u)=cP(u)$$

where $c=T(0)$. Then by the hypothesis (1)

$$cP(f*g)=c^2Pf\cdot P(g).$$

On the other hand by the property (2) we can prove

$$P(f*g)=P(f)P(g).$$

Comparing these formulae

$$c=c^2.$$

Because T is not constant 0, we get

$$c=1$$

and

$$T=P.$$

But by the hypothesis T is bounded, so we may claim $T\in\hat{R}$.

§ 2. In this section we shall assume the followings.

(A) I, J are discrete subrings of R with countable elements.

(B) $T\in\hat{R}$ and $T(xy)=T(yx)$ for x, y in R.

(C) For any open subset U of R and any element a belonging to U there exists a function $f(x)$ in $\mathfrak{D}^0$ whose carrier is contained in U, $\sum_{n\in J} Tf(n)$ is absolutely convergent and $f(a)\neq 0$.[*)]

(D) For any function which appears in the condition (C)

$$\sum_{n\in J} Tf(n)=\sum_{n\in I} F(n).$$

Applying (D) to the function $T(mx)f(x)$ where $m\in J$, we have

$$\sum_{n\in I} T(mn)f(n)=\sum_{n\in J} Tf(n+m)=\sum_{n\in J} Tf(n)=\sum_{n\in I} f(n).$$

If we choose as f such a function that the intersection of Car(f) with I is $\{n_0\}$ only and $f(n_0)\neq 0$, then $T(mn_0)=1$. Thus

Lemma 1. *$T(mn)=1$ if $n\in I$, $m\in J$.*

Conversely we can prove the following

Lemma 2. *If $T(mu)=1$ for all $m\in J$ then $u\in I$.*

[*)] In (C) we may replace "open subset of R" by "neighbourhood of 0", for if f satisfies the condition (C) then f_u satisfies the same condition.

Proof. $\sum_{n\in I} f(n+u)=\sum_{n\in I} f_u(n)$

$$=\sum_{m\in J} Tf_u(n)=\sum_{m\in J} T(-nu)Tf(n)$$
$$=\sum_{m\in J} Tf(n)=\sum_{n\in I} f(n).$$

In a similar manner as in the proof of Lemma 1 we have $u\in I$.

Now we consider the homomorphism ρ from R^+ into $\hat{R}$:

$$\rho: R^+\to\hat{R}$$
$$u\to T(ux).$$

Then by Lemmas 1 and 2 the dual group of $\rho(J)$ is R/I, and consequently R/I is compact.

Proposition 2. *If we assume besides the conditions* (A), (B), (C), (D),

(E) $J=I$ *and*

(F) ρ *is one-to-one correspondence,*

then R^+ *is isomorphic to its dual group and* R/I *is isomorphic to the dual group of* I.

Proof. By the duality theorem $\rho(I)$ is the dual group of R/I, in other words I is the dual group of $\rho(R)/\rho(I)$. On the other hand I is the dual group of $\hat{R}/\rho(I)$. It means that $\hat{R}=\rho(R)$.

To decide whether ρ is one-to-one or not, the following lemma is useful.

Lemma 3. *If* R *satisfies the conditions* (A), (B), (C) *and* (D), *then the kernel of* ρ *is an ideal of* R *and* I *with finite number of elements.*

Proof. Let us denote the kernel with H. Then $T(hn)=1$ for all $n\in J$. Therefore $h\in I$. Clearly H is an ideal of I and R. On the other hand our hypothesis shows

$$\sum_{n\in I} f(n)=\sum_{n\in J} Tf(n)$$
$$=(H)\sum_{N\in\rho(J)}\hat{f}(N)$$

where $\hat{f}$ is the Fourier transform of f. If $(H)=\infty$, then $\sum_{N\in\rho(J)}\hat{f}(N)=0$. Since $\sum_{n\in J} Tf(n)$ is absolutely convergent we get $\sum_{n\in J} Tf(n)=0$ and so $\sum_{n\in I} f(n)=0$. But it is incompatible with the condition (C). Therefore $(H)<\infty$.

Corollary. *If* R *has no element with finite order without* 0 *and satisfies the conditions* (A), (B), (C), (D), *then* ρ *gives a one-to-one correspondence from* R^+ *to* $\hat{R}$.

§ 3. In this section we shall have a consequence of the precedings. From now we shall assume besides (A), (B), (C), (D), (E), (F) that I contains 1 (Condition (G)).

Proposition 3. *If a function* $S(x)$ *defined on* R *satisfies the* *same conditions as $T(x)$, then there exists a unit element e of I such that $S(x)=T(ex)$.*

Proof. Since S belongs to $\hat{R}$ there exists an element e of R such that $S(x)=T(ex)$, and interchanging the roles of S and T we have also $T(x)=S(e'x)$ with $e'\in R$. From these formulae

$$T(x)=T(ee'x),$$

therefore $1=ee'$, because the mapping ρ is isomorphism. By the hypothesis (D) for S and T,

$$\begin{aligned}\sum_{n\in I} f(n)&=\sum_{n\in I} Tf(n)\\ &=\sum_{n\in I} Sf^e(n)v(e)\\ &=\sum_{n\in I} f(ne)v(e)\end{aligned}$$

where $f^e(x)=f(xe)$. If $I\neq Ie$, with suitable choice of f we arrive at a contradiction. So $I=Ie=Ie'$. This means e and e' are units of I.

References

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