21. On the Extension Theorem of the Galois Theory for Finite Factors

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1. We have shown that the fundamental theorem of the Galois theory remains true for finite factors [3] as same as for simple Noetherian rings. Subsequently, in this note, we shall discuss about the so-called extension theorem[1] for finite factors.

We denote by A a continuous finite factor standardly acting on a separable Hilbert space H and by G a finite group of outer automorphisms of A. Put B the set of all elements invariant by G. B is a subfactor of A. Now let C and D be two intermediate subfactors between A and B, then by the fundamental theorem of the Galois theory, there correspond the Galois groups E and F for C and D respectively. That is, E and F are subgroups of G by which C and D are shown as the sets of elements invariant by E and F respectively. Then we may give the extension theorem in the following form.

THEOREM. *Let σ be an isomorphism between C and D fixing every elements of B, then σ may be always extended to an automorphism of A which belongs to G.*

2. We shall begin with some preliminaries. By A^θ we mean the set A equipped with the inner product $\langle a^\theta \mid b^\theta \rangle = \tau(ab^*)$ defined by the standard trace τ of A. As well known, A is faithfully represented on the completion Hilbert space of A^θ. The representation is spatially isomorphic to A acting on H, whence we may identify the representation with A and so A^θ with a dense subset of H. Thus $1^\theta \in H$ gives a trace element of A. The subspace $[1^\theta C]$[2] of H belongs to C'. Since $C' \subset B'$ it belongs B' too. Hence its relative dimension $\dim_{B'}[1^\theta C]$ with respect to B' is meaningful.

As well known, the automorphism group G permits a unitary representation $\{u_g\}$ on H such that $x^g = u_g^* x u_g$ for $x \in A$. Furthermore, as shown in [3], putting $x'^g = u_g^* x' u_g$ for $x' \in A'$, G can be seen as a group of outer automorphisms of A'. Hence we may construct the crossed product $G \otimes A'$ of A' by G, cf. [2]. This can be understand as a von Neumann algebra acting on a Hilbert space $\boldsymbol{H}$ composed of all functions defined on G taking values in H. We show by $\sum_g g \otimes \varphi_g$ a function belonging to $\boldsymbol{H}$ which takes value φ_g at $g \in G$. Then $a' \in A'$

1) Refer to [5] for the theorem of rings with the minimum condition.

2) $[1^\theta C]$ means the metric closure of the set $\{1^\theta c \mid c \in C\}$.

and $g_0 \in G$ define operators $a'^{\#}$ and $g_0^{\#}$ on $\boldsymbol{H}$ respectively such that

$$(\sum_g g\otimes\varphi_g)a'^{\#}=\sum_g g\otimes\varphi_g a', \quad (\sum_g g\otimes\varphi_g)g_0^{\#}=\sum_g gg_0\otimes\varphi_g u_{g_0}.$$

Then the crossed product $G\otimes A'$ is isomorphic to the factor $\boldsymbol{B}'$ generated by $\{a'^{\#}\mid a'\in A'\}$ and $\{g_0^{\#}\mid g_0\in G\}$. It is not hard to see that $\boldsymbol{B}'$ acts standardly on $\boldsymbol{H}$ and its commutor $\boldsymbol{B}$ is generated by $\{a^{\flat}\mid a\in A\}$ and $\{g_0^{\flat}\mid g_0\in G\}$ such that

$$(\sum_g g\otimes\varphi_g)a^{\flat}=\sum_g g\otimes\varphi_g a^g, \quad (\sum_g g\otimes\varphi_g)g_0^{\flat}=\sum_g g_0^{-1}g\otimes\varphi_g, \text{ (cf. [6]).}$$

In the below we show $\{a'^{\#}\mid a'\in A'\}$ and $\{a^{\flat}\mid a\in A\}$ by $\boldsymbol{A}'$ and $\boldsymbol{A}$ respectively.

3. LEMMA 1. *$\dim_{B'}[1^{\mathfrak{e}}C]=1/m$ where m is the order of the group E.*

Proof. We have shown in [3: Lemma 6] that the restriction of $\boldsymbol{B}'$ on a subspace of $\boldsymbol{H}$ having a relative dimension $1/n$ (n is the order of the group G) with respect to the commutor $\boldsymbol{B}$ of $\boldsymbol{B}'$ is spatially isomorphic to the commutor B' of B acting on H.

Since $\boldsymbol{B}'$ acts standardly on $\boldsymbol{H}$, by the above notice and [1: p. 282, Prop. 2] we get $\dim_{B'}[1^{\mathfrak{e}}B]=(1/n)\dim_B[1^{\mathfrak{e}}B']$. Since $[1^{\mathfrak{e}}B']=H$, $\dim_B[1^{\mathfrak{e}}B']=1$. Therefore $\dim_{B'}[1^{\mathfrak{e}}B]=1/n$. Similarly $\dim_{C'}[1^{\mathfrak{e}}C]=1/m$. As $C'\subset B'$,

$$\dim_{B'}[1^{\mathfrak{e}}C]=\dim_{C'}[1^{\mathfrak{e}}C]=1/m. \qquad \text{q.e.d.}$$

Analogously, for D, $\dim_{B'}[1^{\mathfrak{e}}D]=1/m'$, where m' is the order of F.

LEMMA 2. *If there exists an isomorphism σ between C and D such as stated in the theorem, $[1^{\mathfrak{e}}C]$ is equivalent to $[1^{\mathfrak{e}}D]$ with respect to B', that is, $m=m'$.*

Proof. If we put $(1^{\mathfrak{e}}c)\bar{v}_\sigma=1^{\mathfrak{e}}c^\sigma$ for $c\in C$, since by the definition of the inner product of $A^{\mathfrak{e}}$,

$$\langle 1^{\mathfrak{e}}c\mid 1^{\mathfrak{e}}c_1\rangle=\tau(cc_1^*), \quad \langle 1^{\mathfrak{e}}c^\sigma\mid 1^{\mathfrak{e}}c_1^\sigma\rangle=\tau(c^\sigma c_1^{\sigma *})=\tau(cc_1^*),$$

whence $\bar{v}_\sigma$ gives an isometric linear mapping from $[1^{\mathfrak{e}}C]$ onto $[1^{\mathfrak{e}}D]$. Now denote by $[1^{\mathfrak{e}}C]^\perp$ the ortho-complement of $[1^{\mathfrak{e}}C]$. Then every $\varphi\in H$ is decomposed into $\varphi=\varphi_0+\varphi_\perp$ where $\varphi_0\in[1^{\mathfrak{e}}C]$, $\varphi_\perp\in[1^{\mathfrak{e}}C]^\perp$. We define v_σ by $\varphi v_\sigma=\varphi_0\bar{v}_\sigma$, then v_σ is a partial isometric operator defined on H having the initial domain $[1^{\mathfrak{e}}C]$ and the range $[1^{\mathfrak{e}}D]$.

Next we show $v_\sigma\in B'$. Denote by ε the conditional expectation conditioned by C in the sense of Umegaki [7], which projects A onto C. Then $a^{\mathfrak{e}}=a^{\varepsilon\mathfrak{e}}+a_\perp$, where $a_\perp\in[1^{\mathfrak{e}}C]^\perp$ for $a\in A$. Since $a^\varepsilon\in C$, we have

$$a^{\mathfrak{e}}v_\sigma=a^{\varepsilon\mathfrak{e}}\bar{v}_\sigma=a^{\varepsilon\sigma\mathfrak{e}}.$$

For $b\in B$,

$$a^{\mathfrak{e}}v_\sigma b=a^{\varepsilon\sigma\mathfrak{e}}b=(a^{\varepsilon\sigma}b)^{\mathfrak{e}}=(a^\varepsilon b)^{\sigma\mathfrak{e}}.$$

On the other hand we have

$$a^{\mathfrak{e}}bv_\sigma=(ab)^{\mathfrak{e}}v_\sigma=(ab)^{\varepsilon\mathfrak{e}}\bar{v}_\sigma=(a^\varepsilon b)^{\mathfrak{e}}\bar{v}_\sigma=(a^\varepsilon b)^{\sigma\mathfrak{e}}.$$

Since $A^{\mathfrak{e}}$ is dense in H, we get $v_\sigma b=bv_\sigma$ i.e. $v_\sigma\in B'$. q.e.d.

By Lemma 2 we know that there exist trace elements φ_i and ψ_i ($i=1,2,\cdots,m$) of C and D respectively in H, by which H decomposes orthogonally into such as

$$H=[\varphi_1 C]\oplus[\varphi_2 C]\oplus\cdots\oplus[\varphi_m C]=[\psi_1 D]\oplus[\psi_2 D]\oplus\cdots\oplus[\psi_m D].$$

In this case we may assume $\varphi_1=\psi_1=1^\theta$. Putting $(\varphi_i c)u_\sigma=\psi_i c^\sigma$ for $c\in C$ ($i=1,2,\cdots,m$), we get a unitary operator u_σ on H.

LEMMA 3. *$u_\sigma c^\sigma=cu_\sigma$ for every $c\in C$.*

In fact, for $\varphi_i x$ ($x\in C$),

$$\varphi_i x u_\sigma c^\sigma=\psi_i x^\sigma c^\sigma=\psi_i(xc)^\sigma=\varphi_i xcu_\sigma.$$

As σ fixes every element of B, $u_\sigma b=bu_\sigma$, that is, $u_\sigma\in B'$. Now let N be the set of all elements of B' satisfying $yc^\sigma=cy$ for every $c\in C$. By Lemma 3, $u_\sigma\in N$.

LEMMA 4. *$N=C'u_\sigma=u_\sigma D'$.*

Proof. For $y\in N$, $yc^\sigma=cy$ implies $cyu_\sigma^*=yc^\sigma u_\sigma^*=yu_\sigma^*c$, whence $Nu_\sigma^*\subset C'$, that is, $N\subset C'u_\sigma$. Conversely, for $z\in C'$, $czu_\sigma=zcu_\sigma=zu_\sigma c^\sigma$ means $C'u_\sigma\subset N$. Hence we get $N=C'u_\sigma$. On the other hand we have

$$c^\sigma u_\sigma^* c' u_\sigma=u_\sigma^* cc' u_\sigma=u_\sigma^* c' c u_\sigma=u_\sigma^* c' u_\sigma c^\sigma.$$

This means $u_\sigma^* C' u_\sigma\subset D'$, i.e. $C'u_\sigma\subset u_\sigma D'$. By a similar calculation, $u_\sigma D'\subset C'u_\sigma$. Thus we get $N=C'u_\sigma=u_\sigma D'$.

4. LEMMA 5. *Any non-trivial subspace of $\boldsymbol{H}$ does not reduce every element of $\boldsymbol{B}'$ and $\boldsymbol{A}$.*

We say this fact briefly *$\boldsymbol{H}$ is irreducible with respect to $\boldsymbol{A}$ and $\boldsymbol{B}'$.* This lemma is derived from the proof of [2: Theorem 1].

Since $\boldsymbol{B}'$ is algebraically isomorphic to B', there is a subfactor $\boldsymbol{D}'$ of $\boldsymbol{B}'$, which is isomorphic to D'. Hence $\boldsymbol{D}'$ is algebraically isomorphic to the crossed product $F\otimes A'$ of A' by F and it is generated by $\boldsymbol{A}'$ and $\{f^\#\mid f\in F\}$ on $\boldsymbol{H}$ (cf. [2: Theorem 2]. The group G is decomposed by the subgroup F into mutually disjoint cosets $G=g_0F\smile g_1F\smile\cdots\smile g_lF$ where $l=n/m-1$ and $g_0=1$. We show by $\boldsymbol{K}_i$ the subspace of $\boldsymbol{H}$ composed of all functions which vanishes on whole G except a coset g_iF. Then, corresponding to the decomposition of G, $\boldsymbol{H}$ decomposes into mutually orthogonal subspaces as follows: $\boldsymbol{H}=\boldsymbol{K}_0\oplus\boldsymbol{K}_1\oplus\cdots\oplus\boldsymbol{K}_l$. Especially $\boldsymbol{K}_0$ is identified with the space of all functions defined on F taking values in H and so $\boldsymbol{D}'$ acts standardly on $\boldsymbol{K}_0$. Hence $\boldsymbol{K}_0$ is irreducible with respect to $\boldsymbol{A}$ and $\boldsymbol{D}'$ by Lemma 5. Furthermore we get

LEMMA 6. *Every $\boldsymbol{K}_i$ is irreducible with respect to $\boldsymbol{A}$ and $\boldsymbol{D}'$.*

Proof. g_i^{-1b} is a unitary operator belonging to $\boldsymbol{B}$ and it satisfies $\boldsymbol{K}_0 g_i^{-1b}=\boldsymbol{K}_i$ and $g_i^{-1b}a^b=a^{\rho_i b}g_i^{-1b}$. Hence a subspace V of $\boldsymbol{K}_i$ reduces every element of $\boldsymbol{A}$ and $\boldsymbol{D}'$ if and only if a subspace Vg_i^b of $\boldsymbol{K}_0$ has the same property. Thus the irreducibility of $\boldsymbol{K}_0$ leads to that of $\boldsymbol{K}_i$.

Let $\boldsymbol{N}$ be the image of N by the isomorphism of B' onto $\boldsymbol{B}'$ and u_σ be the operator corresponding to $u_\sigma\in B'$ defined in §3 by this isomorphism. Put $\boldsymbol{N}^\theta=[(1\otimes 1^\theta)\boldsymbol{N}]$[3] and $\boldsymbol{C}'^\theta=[(1\otimes 1^\theta)\boldsymbol{C}']$.

LEMMA 7. *$\boldsymbol{N}^\theta$ is irreducible with respect to $\boldsymbol{A}$ and $\boldsymbol{D}'$.*

3) $1\otimes 1^\theta$ means the function $\sum_g g\otimes\varphi_g$ such that $\varphi_g=0$ for $g\neq 1$ and $\varphi_1=1^\theta$.

Proof. As $N=u_\sigma D'$, $\boldsymbol{N}^\theta \supset \mathfrak{u}_\sigma^\theta \boldsymbol{D}'$, whence $\boldsymbol{N}^\theta$ reduces every element of $\boldsymbol{D}'$. $N=C'u_\sigma$ implies $\boldsymbol{N}^\theta \subset [\boldsymbol{C}'^\theta \mathfrak{u}_\sigma]$ and so $[\boldsymbol{N}^\theta \boldsymbol{A}]=[\boldsymbol{C}'^\theta \mathfrak{u}_\sigma \boldsymbol{A}]=[\boldsymbol{C}'^\theta \boldsymbol{A}\mathfrak{u}_\sigma]\subset[\boldsymbol{C}'^\theta \mathfrak{u}_\sigma]=\boldsymbol{N}^\theta$ because $\boldsymbol{C}^\theta$ reduces every element of $\boldsymbol{A}$. Hence $\boldsymbol{N}^\theta$ is invariant by $\boldsymbol{A}$ and $\boldsymbol{D}'$. Since, for a given $d'\in D'$, there is a $c'\in C'$ such that $d'u_\sigma^*=u_\sigma^*c'$, a subspace of V of $\boldsymbol{N}^\theta$ is invariant by $\boldsymbol{A}$ and $\boldsymbol{D}'$ if and only if a subspace Vu_σ^* of $\boldsymbol{C}'^\theta$ is invariant by $\boldsymbol{A}$ and $\boldsymbol{C}'$. By Lemma 5, $\boldsymbol{C}'^\theta$, on which $\boldsymbol{C}'$ acts standardly, is irreducible with respect to $\boldsymbol{A}$ and $\boldsymbol{C}'$. Therefore $V=0$ otherwise $V=\boldsymbol{N}^\theta$.

LEMMA 8. *Let p_i be the projection from $\boldsymbol{H}$ onto $\boldsymbol{K}_i$, then $\boldsymbol{N}^\theta p_i=0$ or $\boldsymbol{N}^\theta p_i=\boldsymbol{K}_i$.*

Proof. Clearly p_i commutes with elements of $\boldsymbol{D}'$. Furthermore
$$(\textstyle\sum_{i,f} g_i f\otimes\varphi)a^\flat p_i=\sum_f g_i f\otimes\varphi a^{g_i\flat}=(\sum_{i,f} g_i f\otimes\varphi)p_i a^\flat,$$
that is, p_i commutes with elements of $\boldsymbol{A}$. Hence $\boldsymbol{N}^\theta p_i$ is a subspace of $\boldsymbol{K}_i$ invariant by $\boldsymbol{A}$ and $\boldsymbol{D}'$. By Lemma 6, $\boldsymbol{K}_i$ is irreducible with respect to $\boldsymbol{A}$ and $\boldsymbol{D}'$ and so $\boldsymbol{N}^\theta p_i=0$ or $\boldsymbol{N}^\theta p_i=\boldsymbol{K}_i$.

LEMMA 9. *If $\boldsymbol{N}^\theta p_i=\boldsymbol{K}_i$, p_i gives a one-to-one bicontinuous mapping of $\boldsymbol{N}^\theta$ onto $\boldsymbol{K}_i$.*

Proof. Since the kernel of p_i is a subspace invariant by $\boldsymbol{A}$ and $\boldsymbol{D}'$, its intersection with $\boldsymbol{N}^\theta$ is 0 or $\boldsymbol{N}^\theta$ itself. By the assumption $\boldsymbol{N}^\theta p_i=\boldsymbol{K}_i$, $\boldsymbol{N}^\theta$ is not in the kernel. Thus p_i is one-to-one. The continuity of the inverse mapping p_i^{-1} follows from the well-known theorem of Banach space.

To simplify the notations, we denote by x_N and $x_{(i)}$ the restriction of x on $\boldsymbol{N}^\theta$ and $\boldsymbol{K}_i$ respectively. Then, as seen from the proof of Lemma 8, we get
$$\varphi p_i^{-1}a'^{\#}_N p_i=\varphi a'^{\#}_{(i)},\quad \varphi p_i^{-1}f^{\#}_N p_i=\varphi f^{\#}_{(i)},\quad \varphi p_i^{-1}a^\flat_N p_i=\varphi a^\flat_{(i)}$$
for $\varphi\in K_i$. $g_i^\flat$ maps $\boldsymbol{K}_i$ isometrically onto $\boldsymbol{K}_0$ and by the definitions of operators $a'^{\#}, a^\flat, f^{\#}$, we get
$$\varphi g_i^{-1\flat}a'^{\#}_{(i)}g_i^\flat=\varphi a'^{\#}_{(0)},\quad \varphi g_i^{-1\flat}f^{\#}_{(i)}g_i^\flat=\varphi f^{\#}_{(0)},\quad \varphi g_i^{-1\flat}a^\flat_{(i)}g_i^\flat=\varphi a^{g_i\flat}_{(0)}$$
for $\varphi\in\boldsymbol{K}_0$.

LEMMA 10. *There exists a $\boldsymbol{K}_i$ such that $\boldsymbol{N}^\theta=\boldsymbol{K}_i$.*

Proof. If there exist $p_i, p_j (i\neq j)$ such that $\boldsymbol{N}^\theta p_i\neq 0$, $\boldsymbol{N}^\theta p_j\neq 0$, we put $\varphi t=\varphi g_i^{-1\flat}p_i^{-1}p_j g_j^\flat$ for $\varphi\in\boldsymbol{K}_0$. t maps $\boldsymbol{K}_0$ into itself and, by the relations stated before the lemma, it commutes with elements of $\boldsymbol{D}'_{(0)}$ and satisfies
$$a^{g_i\flat}_{(0)}t=ta^{g_j\flat}_{(0)}.$$
Since t is in the commutor of $\boldsymbol{D}'_{(0)}$, it permits an expression such that
$$\varphi t=\varphi\textstyle\sum_f (f^\flat a_f^\flat)\quad\text{for}\quad \varphi\in\boldsymbol{K}_0.$$
Hence, as operators defined on $\boldsymbol{K}_0$, we get
$$\textstyle\sum_f (f^\flat a_f^\flat)a^{g_i\flat}=a^{g_j\flat}\sum_f f^\flat a_f^\flat \quad\text{i.e.}\quad \sum_f f^\flat(a_f a^{g_i})^\flat=\sum_f f^\flat(a^{g_j f}a_f)^\flat.$$
This means $a_f a^{g_i}=a^{g_j f}a_f$. Since $g_i\notin g_j F$ and G is outer, $a_f=0$ and so $t=0$ by [2: Lemma 1]. This is a contradiction. Hence, there is only one p_i such that $\boldsymbol{N}^\theta p_i\neq 0$. In other words, $\boldsymbol{N}^\theta\subset\boldsymbol{K}_i$. By the irreducibility

of K_i with respect to A and D', $N^\sigma = K_i$. q.e.d.

5. Proof of the Theorem. Since $u_\sigma \in B'$, it has an expression such that $u_\sigma = \sum_g u_g a'_g$. On the other hand $u_\sigma \in N$ and by Lemma 10, $u_\sigma^\sigma \in K_i$. Therefore if $g \notin g_i F$, $a'_g = 0$. Hence u_σ has an expression such that

$$u_\sigma = u_{g_i}(\textstyle\sum_f u_f a'_{g_i f}) = u_{g_i} d',$$

where $d' = \sum_f u_f a'_{g_i f} \in D'$. d' is a unitary operator and by Lemma 3

$$c^\sigma = d'^* u^*_{g_i} c u_{g_i} d'.$$

Thus

$$u^*_{g_i} c u_{g_i} = d' c^\sigma d'^* = c^\sigma,$$

because $c^\sigma \in D$. This means that the isomorphism σ between C and D coincides with the action of g_i on C and so σ can be extended to the automorphism g_i of A.

REMARK. In the proof of theorem, we have not make any restriction for the choice of a representative g_i from the coset $g_i F$. Therefore we may say, as a version of the theorem, that there corresponds a coset $g_i F$ of G for the isomorphism σ stated in the theorem.

As a consequence of the theorem, we know that G exhausts the automorphisms of A which leave B elementwise fixed. Transferring to the commutors, this means that an inner automorphism of $G \otimes A'$ which preserves A' induces to A' an automorphism belonging to G up to inner automorphisms of A'. This is a theorem shown in the preceding paper [4] restricted within finite groups G.

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