33. Product of Metric Spaces with an Extension Property

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In this note, a *function* is a real-valued uniformly continuous mapping. We say a uniform space has an *extension property* or a *property* E [1] if every function on any uniform subspace of the uniform space has a uniform extension to the whole space. We know [1], [2] some characterisations for a metric space to have the property E. In this note, we are going to find a necessary and sufficient condition for a product space of metric spaces to have the property E. Let V be an entourage in a uniform space, then we shall say that a family of subsets is *V-discrete* if $V(x)$ meets at most one member of the family for every point x of the space, and a family of subsets is *uniformly discrete* if it is V-discrete for some entourage V. $V^\infty=\cup_n V^n$. In a metric space, V_e, e a positive number, is the entourage in the space consisting of pairs of points whose distances are less than e.

Let us recall some known results for later use.

Theorem 1 ([1], Theorem 2). *A pseudo-metric complete space has the property E if and only if, for any positive number e, there is a compact subset K such that, for any open subset G containing K, there is a positive number e' satisfying $V_e(p)\supset V_{e'}^\infty(p)$ for every point $p\notin G$.*

Theorem 2 [1]. *A uniform space has the property E if for any entourage V there is a precompact subset K such that for any open subset G containing K there is an entourage W satisfying $V(p)\supset W^\infty(p)$ for every point $p\notin G$.*

We shall call the following property of a metric space the *property* (*).

(*) For any positive number e there is a positive number e' such that $V_e(p)\supset V_{e'}^\infty(p)$ for every point p in the space.

Theorem 3. *A product space S of complete metric spaces S_α has the property E if and only if*

(1) *S is compact, or*

(2) *all S_α have the property (*), or*

(3) *all but one, say S_β, of S_α have the property (*) and are compact, and S_β has the property E.*

Proof. Suppose that S has the property E and is neither of the type (1) nor (2). Since we can consider S_α as a uniform subspace of S, every S_α has the property E. Some S_β is not compact, which contains a V_e-discrete sequence $\{p_n\}$ of points for some $e>0$. If another S_γ has not the property (*), then it contains a sequence $\{q_n\}$ of points satisfying $V_{e'}(q_n) \not\supset V_{1/n}^\infty(q_n)$ for some $e'>0$ and every n. As a uniform subspace of S, $S_\beta \times S_\gamma$ has the property E. On the other hand, since $\{x_n=(p_n, q_n);\ n=1, 2, \cdots\}$ is uniformly discrete in $S_\beta \times S_\gamma$, any compact subset K in $S_\beta \times S_\gamma$ does not include x_n for all n greater than some n_0, and there is an open subset containing K and disjoint from $\{x_n;\ n>n_0\}$. However, there is no entourage V in $S_\beta \times S_\gamma$ satisfying $V_e \times V_{e'}(x_n) \supset V^\infty(x_n)$, $n>n_0$, which contradicts Theorem 1. Therefore all but S_β have the property (*), and, since S is not (2), S_β has not the property (*), and so all but S_β must be compact as we have proved just above. Conversely, if S is a space of the type (1), (2), or (3), then S has the property E by Theorems 1 and 2.

References

[1] M. Atsuji: Uniform extension of uniformly continuous functions, Proc. Japan Acad., **37**, 10–13 (1961).

[2] H. H. Corson and J. R. Isbell: Some properties of strong uniformities, Quart. J. Math., **11**, 34–42 (1960).