54. *On the Definition of the Knot Matrix*

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The knot matrix defined in [3] for alternating knots can be defined for any knots or links. To do this, we introduce three indices η, d, ε for each crossing point.[1]

Let a regular projection K in S^2 of a knot k have m of the second kind of the loops which divides S^2 into $m+1$ domains, $E_1, E_2, \cdots, E_{m+1}$. Let us denote $(E_i \cup \dot{E}_i) \cap K = K_i$. Then the regions contained in E_i can be classified into two classes, black and white (cf. Lemma 1.10 [3]). For the sake of brevity, any crossing point in K_i is denoted by $c_{i,\lambda}$, any white region in E_i is denoted by $X_{i,\mu}$ and w_i denotes the number of the white regions in E_i.

[Definition 1] For any $c_{i,\lambda}$, the first index η_i is defined as $+1$ or -1 as is shown in the following figure.

$\eta_i = +1$ (a) $\qquad$ $\eta_i = -1$ (b)

(It should be noted that the orientation of k is irrelevant to the definition.)

As usual, two corners among four corners meeting at a crossing point are marked with dots [3].

[Definition 2] For any $c_{j,\lambda}$, the second index d is defined as follows.

(1) $d_{X_{i,\mu}}(c_{j,\lambda}) = 1$ or 0 according as the $c_{j,\lambda}$-corner of $X_{i,\mu}$ is dotted or undotted.

(2) $d_{X_{i,\mu}}(c_{j,\lambda}) = 0$ if $c_{j,\lambda}$ does not lie on $\dot{X}_{i,\mu}$.

If $c_{j,\lambda}$ lies on $\dot{X}_{j,\mu}$, then the third index $\varepsilon_{X_{j,\mu}}(c_{j,\lambda})$ is defined as $+1$ or -1 according as the $c_{j,\lambda}$-corner of $X_{j,\mu}$ is dotted or undotted.

By means of these indices, the knot matrix of K can be defined.

[Definition 3] The knot matrix $M = (M_{ij})_{i,j=1,2,\cdots,m+1}$ is defined as follows:

$$
(1) \qquad
\begin{aligned}
&M_{ii} = (a^{(i)}_{pq})_{p,q=1,\cdots,w_i} \\
&-a^{(i)}_{pq} = \sum_{c_{i,\lambda} \in \dot{X}_{i,p} \cap \dot{X}_{i,q}} \eta_i(c_{i,\lambda})\, d_{X_{i,p}}(c_{i,\lambda}), \quad (p \neq q), \\
&a^{(i)}_{pp} = -\sum_{\substack{p=1 \\ p \neq q}}^{w_i} a_{pq}.
\end{aligned}
$$

1) For symbols and notations, see [3].

$$
\begin{aligned}
&M_{ij}=(b_{rs}^{(ij)})_{r=1,\cdots,w_i,\, s=1,\cdots,w_j} \quad \text{for} \quad i \neq j, \\
(2) \qquad &-b_{rs}^{(ij)}=\sum_{c_{j,\lambda}\in \dot{x}_{i,r}\frown \dot{x}_{j,s}} \eta_i(c_{j,\lambda})d_{x_{i,r}}(c_{j,\lambda})\varepsilon_{x_{j,s}}(c_{j,\lambda}).
\end{aligned}
$$

Then using the same notation as is used in [3], we can prove the following

[Theorem 1] Let $\Delta(t)$ be the A-polynomial. Then

$$\pm t^{\lambda}\Delta(t)=\det\{\widetilde{M}\binom{i_1 i_2\cdots i_{m+1}}{i_1 i_2\cdots i_{m+1}}-t\widetilde{M}^{t}\binom{i_1 i_2\cdots i_{m+1}}{i_1 i_2\cdots i_{m+1}}\}.$$

In general, the knot matrix of a non-alternating knot does not satisfy the following conditions, which are always satisfied for alternating knots,

(1.1) $\qquad a_{pp}^{(i)}a_{pq}^{(i)}\leqq 0,\ a_{pp}^{(i)}a_{qp}^{(i)}\leqq 0,\ |a_{pp}^{(i)}|\geqq|a_{pq}^{(i)}|$, for all p, q.

However, there are some cases which satisfy (1.1). In such cases their genera are calculated exactly [1, 3]. For example, we have

[Theorem 2] Let $k_{p,q}$ be the parallel knot of type (p,q) whose carrier knot is a special alternating knot.[2] Then the genus of k is one half of the degree of its Alexander polynomial. This is especially true for all torus knots.[3]

References

[1] R. H. Crowell: Genus of alternating link types, Ann. of Math., **69**, 259-275 (1959).

[2] K. Murasugi: On the Alexander polynomial of the alternating knot, Osaka Math. J., **10**, 181-189 (1958).

[3] ——: On alternating knots, Osaka Math. J., **12**, 277-303 (1960).

2) $k_{p,q}$ are nonalternating for $p\geqq 3$, q arbitrary [2].

3) This was shown by H. Seifert.