

54. On the Definition of the Knot Matrix

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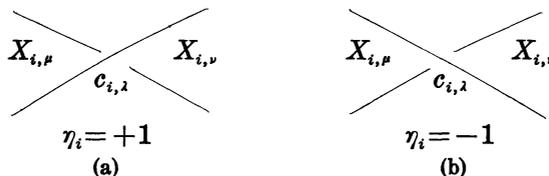
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The knot matrix defined in [3] for alternating knots can be defined for any knots or links. To do this, we introduce three indices η, d, ε for each crossing point.¹⁾

Let a regular projection K in S^2 of a knot k have m of the second kind of the loops which divides S^2 into $m+1$ domains, E_1, E_2, \dots, E_{m+1} . Let us denote $(E_i \setminus \dot{E}_i) \cap K = K_i$. Then the regions contained in E_i can be classified into two classes, black and white (cf. Lemma 1.10 [3]). For the sake of brevity, any crossing point in K_i is denoted by $c_{i,\lambda}$, any white region in E_i is denoted by $X_{i,\mu}$ and w_i denotes the number of the white regions in E_i .

[Definition 1] For any $c_{i,\lambda}$, the first index η_i is defined as +1 or -1 as is shown in the following figure.



(It should be noted that the orientation of k is irrelevant to the definition.)

As usual, two corners among four corners meeting at a crossing point are marked with dots [3].

[Definition 2] For any $c_{j,\lambda}$, the second index d is defined as follows.

(1) $d_{X_{i,\mu}}(c_{j,\lambda}) = 1$ or 0 according as the $c_{j,\lambda}$ -corner of $X_{i,\mu}$ is dotted or undotted.

(2) $d_{X_{i,\mu}}(c_{j,\lambda}) = 0$ if $c_{j,\lambda}$ does not lie on $\dot{X}_{i,\mu}$.

If $c_{j,\lambda}$ lies on $\dot{X}_{j,\mu}$, then the third index $\varepsilon_{X_{j,\mu}}(c_{j,\lambda})$ is defined as +1 or -1 according as the $c_{j,\lambda}$ -corner of $X_{j,\mu}$ is dotted or undotted.

By means of these indices, the knot matrix of K can be defined.

[Definition 3] The knot matrix $M = (M_{ij})_{i,j=1,2,\dots,m+1}$ is defined as follows:

$$\begin{aligned}
 M_{ii} &= (a_{pq}^{(i)})_{p,q=1,\dots,w_i} \\
 -a_{pq}^{(i)} &= \sum_{c_{i,\lambda} \in \dot{X}_{i,p} \cap \dot{X}_{i,q}} \eta_i(c_{i,\lambda}) d_{X_{i,p}}(c_{i,\lambda}), \quad (p \neq q), \\
 a_{pp}^{(i)} &= - \sum_{\substack{p=1 \\ p \neq q}}^{w_i} a_{pq}^{(i)}.
 \end{aligned}
 \tag{1}$$

1) For symbols and notations, see [3].

$$(2) \quad \begin{aligned} M_{ij} &= (b_{rs}^{(ij)})_{r=1, \dots, w_i, s=1, \dots, w_j} \quad \text{for } i \neq j, \\ -b_{rs}^{(ij)} &= \sum_{c_{j,\lambda} \in \tilde{X}_{i,r} \cap \tilde{X}_{j,s}} \eta_i(c_{j,\lambda}) \delta_{X_{i,r}}(c_{j,\lambda}) \varepsilon_{X_{j,s}}(c_{j,\lambda}). \end{aligned}$$

Then using the same notation as is used in [3], we can prove the following

[Theorem 1] Let $\Delta(t)$ be the A -polynomial. Then

$$\pm t^i \Delta(t) = \det \{ \tilde{M}_{(i_1 i_2 \dots i_{m+1})}^{(i_1 i_2 \dots i_{m+1})} - t \tilde{M}'_{(i_1 i_2 \dots i_{m+1})} \}.$$

In general, the knot matrix of a non-alternating knot does not satisfy the following conditions, which are always satisfied for alternating knots,

$$(1.1) \quad a_{pp}^{(i)} a_{pq}^{(i)} \leq 0, \quad a_{pp}^{(i)} a_{qp}^{(i)} \leq 0, \quad |a_{pp}^{(i)}| \geq |a_{pq}^{(i)}|, \quad \text{for all } p, q.$$

However, there are some cases which satisfy (1.1). In such cases their genera are calculated exactly [1, 3]. For example, we have

[Theorem 2] Let $k_{p,q}$ be the parallel knot of type (p, q) whose carrier knot is a special alternating knot.²⁾ Then the genus of k is one half of the degree of its Alexander polynomial. This is especially true for all torus knots.³⁾

References

- [1] R. H. Crowell: Genus of alternating link types, *Ann. of Math.*, **69**, 259-275 (1959).
- [2] K. Murasugi: On the Alexander polynomial of the alternating knot, *Osaka Math. J.*, **10**, 181-189 (1958).
- [3] —: On alternating knots, *Osaka Math. J.*, **12**, 277-303 (1960).

2) $k_{p,q}$ are nonalternating for $p \geq 3$, q arbitrary [2].

3) This was shown by H. Seifert.