107. *Inverse Images of Closed Mappings. III*

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K. Nagami has recently obtained the following theorem:[1] *a completely regular T_1-space X is compact if and only if the projection from the product space $X\times Y$ onto Y is closed for any completely regular T_1-space Y.*

In this note, with the exception of the complete regularity and the separation axiom (T_1) of X, we shall prove an analogous theorem.

Theorem. *Let X be a topological space and $\mathfrak{m}$ an infinite cardinal number. Then X is $\mathfrak{m}$-compact if and only if the projection from the product space $X\times Y$ onto Y is closed for any paracompact Hausdorff space Y such that each point of Y has a neighborhood basis of power $\leqq\mathfrak{m}$.*

Proof. As the "only if" part has been shown in our previous note,[2] we need only prove the "if" part. If we suppose that X is not $\mathfrak{m}$-compact, then there exists a collection of closed subsets $\mathfrak{F}=\{F_\lambda\mid\lambda\in\Lambda\}$ with the finite intersection property such that

(1) $|\Lambda|\leqq\mathfrak{m}$ where $|\Lambda|$ denotes the power of Λ.

(2) $\bigcap_{\lambda\in\Lambda}F_\lambda=\phi$.

Moreover, by adding to $\mathfrak{F}$ all the intersections of finitely many sets of $\mathfrak{F}$, we can assume that $\mathfrak{F}$ satisfies the following condition (3), because $|\Lambda|$ does not exceed $\mathfrak{m}$.

(3) $F_\lambda\frown F_\mu\in\mathfrak{F}$ for any two sets F_λ, F_μ of $\mathfrak{F}$.

We define the partial order in such a way that $\lambda\geqq\mu$ if and only if $F_\lambda\subset F_\mu$. Then Λ is a directed set by the condition (3).

Let Y denote the set of different elements $\{y_\lambda\mid\lambda\in\Lambda\}\smile y_\infty$, where $\infty\neq\lambda$ for every $\lambda\in\Lambda$. We next define the topology of Y such that

(4) the neighborhood basis of each point y_λ is the single point set $\{y_\lambda\}$,

(5) the neighborhood basis of the point y_∞ is the family of sets $U_\lambda(y_\infty)=\{y_\mu\mid\mu\geqq\lambda\}\smile y_\infty$.

Then, since Λ is a directed set, Y is a topological space. It is evident that each point of Y has a neighborhood basis of power $\leqq\mathfrak{m}$. We next prove that Y is a Hausdorff space. Since $\{y_\lambda\}\frown\{y_\mu\}=\phi$

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1) K. Nagami communicated to me this interesting theorem in his kind letter of August 8, 1961.

2) S. Hanai: Inverse images of closed mappings. I, Proc. Japan Acad., **37**, 298–301 (1961).

for $\lambda \neq \mu$, it is sufficient to show that there exist disjoint neighborhoods of two points y_λ and y_∞. From the condition (2), we can see that there exists a set F_μ such that $F_\lambda \supsetneqq F_\mu$. Then $\{y_\lambda\} \frown U_\mu(y_\infty) = \phi$, hence Y is a Hausdorff space. It is easy to see that, for any open covering $\mathfrak{G}$ of Y, we can find a locally finite open refinement of $\mathfrak{G}$ such that $\{U_\lambda(y_\infty); \{y_\mu\} \mid y_\mu \in Y - U_\lambda(y_\infty)\}$ where λ is a suitable suffix. Therefore Y is para-compact.

Let $\mathfrak{L} = \{F_\lambda \times y_\lambda \mid \lambda \in \Lambda\}$, then $\mathfrak{L}$ is a family of closed subsets of the product space $X \times Y$. We shall next prove that $\mathfrak{L}$ is locally finite. Let $U(x) \times \{y_\lambda\}$ be a neighborhood of the point (x, y_λ), then $U(x) \times \{y_\lambda\}$ intersects only one element $F_\lambda \times y_\lambda$ of $\mathfrak{L}$. Let x be any point of X, then there exists a set F_λ such that $x \notin F_\lambda$ since $\bigcap_{\lambda \in \Lambda} F_\lambda = \phi$. Then $(X - F_\lambda) \times U_\lambda(y_\infty)$ is a neighborhood of the point (x, y_∞). Since $\lambda \leqq \mu$ follows from $y_\mu \in U_\lambda(y_\infty)$, we have $F_\mu \subset F_\lambda$. Then $F_\mu \frown (X - F_\lambda) = \phi$. Therefore no element of $\mathfrak{L}$ intersects the neighborhood $(X - F_\lambda) \frown U_\lambda(y_\infty)$. By the above reasoning, we can see that $\mathfrak{L}$ is locally finite. Therefore $A = \bigcup_{\lambda \in \Lambda} (F_\lambda \times y_\lambda)$ is a closed subset of $X \times Y$. It is evident that the projection p from $X \times Y$ onto Y transforms the set A onto the set $C = \bigcup_{\lambda \in \Lambda} \{y_\lambda \mid \lambda \in \Lambda\}$. On the other hand, since y_∞ is a cluster point of C, C is not closed. Hence p is not closed. This completes the proof.

As the immediate consequences of the above theorem, we get the following corollaries.

Corollary 1. *A topological space X is compact if and only if the projection from the product space $X \times Y$ onto Y is closed for any paracompact Hausdorff space Y.*

Corollary 2. *A topological space X is countably compact if and only if the projection from the product space $X \times Y$ onto Y is closed for any paracompact Hausdorff space Y satisfying the first countability axiom.*

Remark. From the proof of the above theorem, we can see that "for any paracompact Hausdorff space Y" may be replaced by "for any non-discrete paracompact Hausdorff space Y". Therefore the proposition replaced by "for any non-discrete paracompact Hausdorff space Y" in Corollary 2 is stronger than Corollary 1.7 in our previous note.[3)]

3) Cf. the note cited in 2).