106. "Foundation" and Formalism

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Has formalism really succeeded in laying the foundations of mathematics? If it has not, in what relation does it stand to foundation? Further, is the conception of foundation a legitimate one? These are the questions which I try to answer in this paper, for I am of the opinion that inquiry into them at the same time affords a clue to the ways of thinking underlying various views of the foundations of mathematics.

First, I take it that foundation is an operation aimed at examining whether an assertion which we make is correct and, if so, explaining why. As seen in this light, its basic attitude is not peculiar to it and mathematics, but is the one that constitutes the essential method of science in general. Only, science makes it its first duty to elucidate the unknown, while foundation is concerned to judge whether what is supposed to be known is really known. In other words, both of them have one and the same attitude in common, although they are interested in different spheres. Hence it may be concluded that the attitude in question is fairly sound.

Historically, too, in the evolution of the foundation of the differential and integral calculus which forms the prehistory of the foundations of mathematics, the method of ascertaining whether a seemingly self-evident matter was really an indisputable fact took the leading rôle, as with Cauchy, Dedekind and Cantor.

This method, however, cannot be taken for granted, because foundation presupposes the concept of the "true", which is thought to be meaningless. Speaking formularily, when the matter in hand is deduceed from what has already been admitted to be true, it is regarded as founded. Now, the concept of the true involves in the last analysis the antiquated rule of "adquaetio rei et intellectus", which in turn postulates that the object has an existence independent from the subject, and that it can be grasped as a pure idea. This view belongs to realism in the epistemological sense and sanctions apriorism of some sort or other with regard to the truth or falsehood of a synthetic judgement. The way of thinking underlying Euclid's Elements is a typical specimen of it. But the possibility of grasping the object as an idea on which this view rests cannot be proved positively. In this respect, the phenomenological view of logical positivists is right, in so far as it holds that in real sciences, where the objects are regarded as objective existences, the knowledge of them is a hypothesis which may probably be true, never can be certainly so. Where the objects are not objective existences, they are merely assumed by us, and then the knowledge of them is absolutely hypothetical. As is widely known, such is the new standpoint of axiomatism in mathematics. In any case, it follows that the absolutely true which was presupposed by "foundation" does not exist anywhere. Indeed, there exist "foundamental principles" which can be applied to various branches of science, but no "foundation" which provides a criterion for judging whether a given theory is true or false. What is at present called the foundations of mathematics is a system of branches of mathematics connected in origin with the department which was once studied with an eye to foundation in the latter sense. This is the prevailing view of today. The ambiguity in one way or another of the sphere of existence of the objects discussed in the afore-said foundation of the late 19th century is perhaps due to the insufficiency of epistemological reflection mentioned above. Such deficiency is examplified in Dedekind's Meine Gedankenwelt, Cantor's "set concept" etc.

The question of truth and falsehood has further to do with logic. According to axiomatism, an individual proposition is a hypothetical one, but an inference is either true or false. (Although axiomatic logic holds an inference also to be neither true nor false, on which point some remarks will be made later on). To pronounce on the truth or the falsehood of an inference is the function of logic. But what character logic as a science should have seemed to be a question rather difficult to answer. To speak plainly, if logic is a science, its theorems ought to be supplied with demonstrations, which consists of reasoning. Here, it is suspected, is a vicious circle, since the reasoning itself should first be put to test. Again, if the axioms of logic are hypothetical, are they not irrelevant to the reasoning actually carried on? On the other hand, the assertion that they are not hypothetical seems to be a relapse to the old apriorism which was refuted by axiomatism.

Symbolic logic, in order to avoid such ambiguity, renders propositions and logical concepts in symbols and further deprives them of their concrete meanings, thus reducing propositions to groups of "Zeichen ohne Zeigen". Next, it lays down rules governing the arrangement and transformation of a group of symbols and then examines how it can be transformed under the rules. In short, it regards logic as a science inquiring into the arrangement and transformation of symbols. Since the rules are laid down by us, logic is necessarily axiomatic and admits of the existence of a variety of systems. From this point of view, an axiomatically organized science, not mathematics alone, is a study of what concrete arrangements of symbols are obtained when some groups of symbols are transformed according to the rules of symbolic logic. Logic is a system of knowledge concerning the transformation of symbols in general and concrete sciences treat of various relations between concrete arrangements of symbols. This difference in function is analogous to that formerly seen between logic and concrete sciences, the respective tasks of which were to clarify general rules of reasoning and to study interrelations between concrete propositions by the use of the rules of logic. In the foregoing pages has been presented an answer given by formalism to the question what precise knowledge is, that is, what science is.

This view is certainly definite, but it does not seem to be quite free from some objections. For instance, does it lay the foundation on the question, whether a reasoning operation actually carried on in conformity with this view is correct or not? Formalism indeed deals with mere meaningless symbols as its objects of speculation, but when actually arguing from its standpoint, it forms various judgements on the arrangement and transformation of the symbols. This speculation is called the "syntax". When we draw concrete inferences in a syntax, formalisitic symbolic logic does not answer for their accuracy. Because logical concepts used in symbolic logic are in reality nothing but symbols, though they seem to be "logical concepts", so far as their names are concerned. I think that this character of formalism is detected in the application of so-called mathematical induction. By mathematical induction is meant in this case the way of reasoning which is called the "recursive" or "constructive" method actually used in a syntax. Therefore, to demonstrate by mathematical induction as a syntax the consistency of the system of axioms on natural numbers including mathematical induction as an object language is no mere vicious circle. But whenever mathematical induction is used as a syntax, it is assumed that counting can be done in natural numbers, and that mathematical induction is applicable to natural numbers. And natural numbers in this case are not those as the objects of the formalized theory of natural numbers. The validity of this assumption seems to be granted by intuition alone. In this light, formalism may be said not to be quite competent in establishing the legitimacy of its syntactical operations. Only this statement is not intended to be a pronouncement on the merit or demerit of formalism. Its achievements cannot be ignored: e.g. the introduction of symbols by symbolic logic, the definite presentation of problems by the use of symbols, the discovery of problems accessible only to its new point of view, etc. Yet, if we persistently demand a proof of correctness, we cannot but be sceptical. But is such a proof possible? If it is, by what standpoint can it be furnished? In the first place, as to the relation between the subject and the object, the object is not what is assumed by us, but, in a way, what is given to us, because the object of speculation is thinking itself or mathematical matters necessarily involved in the thinking itself, and these are facts which we actually experience. Accordingly, we do not assume an object as an axiom. In thinking of it we are analyzing a fact. Then we are taking an attitude resembling that of realism (epistemological) refuted by aximatism. Notwithstanding, we do not merely revert to realism. It was refuted in that it regarded an object as an existence independent from the subject, while the theory now in question holds that an object reveals its existence in the field of a whole including the thinking subject itself or in thinking itself. Assuredly, if we try to say something about such a question, we shall have to resort to the conscious analysis of such facts we actually experience or, to use a more comprehensive term, of thinking itself. But then, can such a procedure produce objective knowledge? Or, is objective science on such a question possible? A variety of views may be held on this issue. Formularily speaking, the subject cannot be made an object: to apply the above-mentioned term, speculation on the syntax requires another syntax, and so the syntax itself will never be comprehended as an object. On the other hand, it seems that insistence on being absolutely convinced inevitably leads to reflection on thinking itself. At this point views diverge on the foundations of mathematics. Whether formalism has succeeded in providing mathematics with foundations, in what relation formalism and foundation stand to each other, these questions may be variously answered according as thinking and knowing are viewed in different way. There is, and need be, no consensus on the foundations of mathematics. Such being the case, I think there is room for a method of examining the point of view itself with due regard to knowing itself besides that of formalism which when once a point of view has been fixed, starts on an exploration of the unknown rather than confirm the presumedly known. If such a method as this may be called "foundation", I believe there is still a sufficient raison d'être of "foundation".

In this connection, the following remark of Poincaré's though it was made a long time ago, and has since been followed by the great development of formalism, holds good, if slight modifications are made in terminology. "In a word, both Mr. Russell and Mr. Hilbert made great endeavours. Each of them wrote an original and profound book which some times contains quite right views. These two books of theirs give us abundant food for reflection and are full of valuable lessons we should learn. Some, nay many, of their achievements are destined to remain in steady and permant existence. However, if one says that these scholars have made a final decision on the controversy between Kant and Leibnitz, and so positively confuted Kant's mathematical theory, one is evidently mistaken. I do not know if the two scholars themselves believed they had done so. If they did. They were mistaken".