3. A Note on Almost Periodic Transformation Groups

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A theorem of Gottschalk and Hedlund [1, page 39][2)] contains the following: X is an almost periodic compact minimal orbit-closure under the transformation group (X, T, π) if and only if there exists a unique group structure of X which makes X a topological group. It is assumed that X is a uniform space and T is Abelian. This very strong result depends on the compactness of X in several respects; however, the existence of a group structure can be proved without assuming that X is compact if T is assumed to be uniformly continuous and X is an orbit under the closure of the transition group in the group of all unimorphisms of X onto X.

We let Φ be the group of unimorphisms of X onto X and note that with the space-index topology Φ is a topological group [1, page 94]. We note also that (X, Φ, ρ), where $(x, \varphi)\rho = x\varphi$, is a topological transformation group since the space-index topology is admissible [1, 11.02]. Since the transition group $\Phi_0 = \{\pi^t \mid t \in T\}$ is an Abelian subgroup of Φ so is $\Psi = \bar{\Phi}_0$.

Under the assumption that T is Abelian and uniformly continuous we have the

Theorem. If (X, T, π) is almost periodic and $X = x\Psi$ or some $x \in X$, then there is a group structure on X which makes X a topological group.

For a compact minimal orbit-closure X, we have Ψ compact hence $x\Psi$ is closed. Thus, since $S = \overline{xT}$, $X = x\Psi$ for each $x \in X$. Therefore above theorem is a generalization of the "only if" part of the theorem of Gottschalk and Hedlund.

We first prove the following:

Lemma. If X is a uniform space, T an Abelian group and (X, T, π) is almost periodic, then T is equicontinuous.

Proof. Let α be any index of X. Take a symmetric index β such that $\beta^3 \subset \alpha$. There are subsets A, K of T such that K is compact, $AK = T$ and $xa \in x\alpha$ for each $a \in A$ and $x \in X$. Since K is compact $\{\pi^t \mid t \in K\}$ is equicontinuous [1, 1.20 part 2, page 4]. Thus, for each $x \in X$, there is a neighborhood V such that $(xk, yk) \in \beta$ whenever $y \in V$

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2) We use the notation of [1] throughout.

and $k \in K$. Now let t be any element of T and y and element of V. There are elements a, k of A and K such that $t=ak$. Thus $(xt, yt) = (xak, yak) = (xka, xk) \cdot (xk, yk) \cdot (yk, yka) \in \beta^3$.

Proof of the theorem. Except for the last step we follow [1, page 39]. We define the continuous function f on Ψ onto X by putting $f(\varphi) = x\varphi$ for each $\varphi \in \Psi$. The continuity of f follows from the fact that the space-index topology is admissible and, f is onto since $x\Psi = X$. Let P be the period of x under Ψ. Since P is a subgroup of Ψ we can form the factor group Ψ/P. Now $\varphi_1\varphi_2^{-1} \in P$ if and only if $x\varphi_1 = x\varphi_2$. Thus each y in X determines uniquely an element $P\varphi$ in Ψ/P such that $\varphi = y$. For y in X we define $yh = P\varphi$ where $x\varphi = y$. It follows that h is a one to one function from X onto Ψ/P. We have $Uh = Uf^{-1}\eta$ for any subset U of X where η is the natural homomorphism of Ψ onto Ψ/P. The function h is open since f is continuous and η is open. We complete the proof by showing that h is continuous. It is sufficient to show continuity of the function $\bar{h}$ defined by $y\bar{h} = \varphi$ where φ is any element of Ψ such that $x\varphi = y$. For then $h = \bar{h}\eta$. Let α be any index of X, $y \in X$ and $\varphi = y\bar{h}$. We obtain a neighborhood V of y which is carried into $U_\alpha = \{\psi \in \Psi \mid (z\varphi, z\psi) \in \alpha$ for all z in $X\}$ under $\bar{h}$. By the lemma Φ_0, hence Ψ, is equicontinuous. Therefore there is a neighborhood V of y such that $(y\psi, v\psi) \in \alpha$ whenever $v \in V$ and $\psi \in \Psi$. Now let v be any point of V and take $\theta = v\bar{h}$, i.e., $x\theta = v$. If $z \in X$, there is ψ in Ψ such that $x\psi = z$. Now $(z\varphi, z\theta) = (x\psi\varphi, x\psi\theta) = (x\varphi\psi, x\theta\psi) = (y\psi, v\psi) \in \alpha$. Thus $V\bar{h} \subset U_\alpha$ as was to be shown.

Reference

[1] W. H. Gottschalk and G. A. Hedlund: Topological dynamics, Amer. Math. Soc. Colloquium Publication, vol. **36** (1955).