18. On a Maximum Principle for Quasi-linear Elliptic Equations

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1. Introduction. In this note we shall consider second order quasi-linear elliptic equations of the form

$$\text{(A)} \qquad \sum_{i,j=1}^{n} a_{ij}(x, u, \operatorname{grad} u)\frac{\partial^2 u}{\partial x_i \partial x_j} = f(x, u, \operatorname{grad} u)^{1),2)}$$

whose solutions $u(x)$ are assumed to exist and to be of class C^2 in some domain G.

The purpose of this paper is to establish a maximum principle for solutions of the equations (A) under comparatively mild assumptions so as to extend the classical maximum principles.[3] Once the maximum principle has been established, our next task is to exhibit some of its applications. Thus, for instance, the uniqueness of the solution of the Dirichlet problem for some quasi-linear elliptic equations will be proved.

We can show, in view of the similarity lying between elliptic and parabolic equations, the validity of an analogous maximum principle for quasi-linear parabolic equations of the second order. However, its description will be left to another opportunity.

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2. Maximum principle. We shall begin with the simplest and the most evident fact concerning the maximum principle for the equation (A).

Proposition. *Let the following conditions be satisfied:*

i) *The quadratic form* $\sum_{i,j=1}^{n} a_{ij}(x, u, 0)\xi_i\xi_j$ *is positive definite for every* x *and* u *under consideration.*

ii) *The function* $f(x, u, 0)$ *is positive for positive* u.

Then any solution $u(x)\in C^2(G)$ *of the equation* (A) *cannot assume its positive maximum in the interior of* G.

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1) $x=(x_1,\cdots,x_n)$ and $\operatorname{grad} u=(\partial u/\partial x_1,\cdots,\partial u/\partial x_n)$.

2) The functions $a_{ij}(x, u, p)$ and $f(x, u, p)$ are defined in some domain in the space $(x, u, p)=(x_1,\cdots,x_n, u, p_1,\cdots,p_n)$.

3) See, e.g., Miranda [3], pp. 3–5.

If, in the above proposition, the condition ii) is replaced by the following

ii′) The function $f(x, u, 0)$ is positive for every u.

then the solution $u(x)$ cannot attain its *maximum* in the interior of G.

Now we state our main result.

Theorem. *We assume that the following conditions are satisfied:*

I) *There exists a positive lower semi-continuous function $h(x, u, p)$ such that*

$$\sum_{i,j=1}^{n} a_{ij}(x, u, p)\xi_i\xi_j \geqq h(x, u, p)\,||\,\xi\,||^2 \ [4]$$

for every x, u, p under consideration and for every real vector ξ.

II) *The function $f(x, u, 0)$ is non-negative for non-negative u.*

III) *$f(x, u, p)-f(x, u', p')\geqq -L(|\,u-u'\,|+||\,p-p'\,||)$ with positive $L=L(\Omega, M, N)$ depending only on Ω, M, and N if x varies in any compact subset Ω of G and $|\,u\,|$, $|\,u'\,|\leqq M$ and $||\,p\,||$, $||\,p'\,||\leqq N$, M and N being any constants.*[5]

We assume further that a solution $u(x)\in C^2(G)$ of the equation (A) *attains its non-negative maximum at some interior point x^0 of G. Then $u(x)$ is reduced to a constant in the component G_0 of the set $\{x\in G; u(x)\leqq u(x^0)=m\}$ containing x^0.*

Proof. We denote by G_1 the set $\{x\in G_0; u(x)=m\}$ and assume that $G_1\neq G_0$. Our aim is to derive a contradiction from this assumption.

Let $S(\subset G_0-G_1)$ be a sphere of radius R and with center at the origin whose boundary has only one point x^1 in common with G_1 and S_1 a sphere of radius less than R and with center at x^1. We define a function $v(x)$ by

$$v(x)=u(x)+\varepsilon v_0(x),$$
$$v_0(x)=\exp(-k\,||\,x\,||^2)-\exp(-kR^2)$$

where ε and k are positive constants. For sufficiently small ε we have $v(x)<u(x^1)=u(x^0)=m$ on the boundary ∂S_1 of S_1 and hence $\max_{\bar{S_1}} v(x)(\geqq m)$ is achieved at an interior point x^2 of S_1. The following relations are obvious:

$$||\,\mathrm{grad}\, u(x^2)\,||=2k\varepsilon\,||\,x^2\,||\exp(-k\,||\,x^2\,||^2), \tag{1}$$
$$|\,u(x^1)-u(x^2)\,|\leqq\varepsilon v_0(x^2)\leqq\varepsilon\exp(-k\,||\,x^2\,||^2). \tag{2}$$

If we denote by $\mathcal{L}$ a linear elliptic partial differential operator defined by

$$\mathcal{L}=\sum_{i,j=1}^{n} a_{ij}(x, u(x), \mathrm{grad}\, u(x))\frac{\partial^2}{\partial x_i\partial x_j}$$

we have for sufficiently large k

$$\mathcal{L}v(x^2)>0. \tag{3}$$

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4) $||\cdot||$ denotes the usual norm of an n-dimensional real vector.

5) We shall often encounter such constants L in the sequel. Their meaning being obvious from the context at any time, no detailed explanation will be given as to them.

In fact, with the aid of the assumptions of the theorem and the relations (1), (2) we see that

$$\begin{aligned}\mathcal{L}v(x^2) &= \mathcal{L}u(x^2)+\varepsilon\mathcal{L}v_0(x^2)\\ &= f(x^2, u(x^2), \operatorname{grad} u(x^2)) - f(x^2, u(x^1), 0) + f(x^2, u(x^1), 0) + \varepsilon\mathcal{L}v_0(x^2)\\ &\geqq -L(|u(x^2)-u(x^1)|+\|\operatorname{grad} u(x^2)\|)+\varepsilon\mathcal{L}v_0(x^2)\\ &\geqq \varepsilon\exp(-k\|x^2\|^2)[4k^2h(x^2, u(x^2), \operatorname{grad} u(x^2))\|x^2\|^2-2kL\|x^2\|\\ &\qquad -2k\sum_{i=1}^{n} a_{ii}(x^2, u(x^2), \operatorname{grad} u(x^2))-L].\end{aligned}$$

Since the coefficient of k^2 in the bracket of the last expression is positive, we are able to choose k so large as to make $\mathcal{L}v(x^2)$ positive.

On the other hand, from the fact that the matrix $\|a_{ij}(x, u, p)\|$ is positive definite and that the function $v(x)$ is maximal at x^2 it follows that

$$\mathcal{L}v(x^2)=\sum_{i,j=1}^{n} a_{ij}(x^2, u(x^2), \operatorname{grad} u(x^2))\frac{\partial^2 v(x^2)}{\partial x_i \partial x_j}\leqq 0$$

which contradicts (3). This desired contradiction proves our theorem.

The following three corollaries are immediate consequences of the main theorem. They need no proofs.

Corollary 1. *Under the same assumptions as in Theorem except that the condition* II) *is replaced by the following*

II′) *The function $f(x, u, 0)$ is non-negative for every u.*

we can conclude that if any solution $u(x)\in C^2(G)$ of (A) *achieves its maximum at some interior point x^0 then $u(x)$ is reduced to a constant in G_0.*

Corollary 2. *Let the following assumptions be fulfilled:*

I) *There exists a positive lower semi-continuous function $h(x, u, p)$ such that*

$$\sum_{i,j=1}^{n} a_{ij}(x, u, p)\xi_i\xi_j \geqq h(x, u, p)\|\xi\|^2.$$

II*) *The function $f(x, u, 0)$ is non-negative for non-negative u and is non-positive for non-positive u.*

III*) $|f(x, u, p)-f(x, u', p')|\leqq L(|u-u'|+\|p-p'\|)$.[6]

Let $u(x)\in C^2(G)\frown C^0(\overline{G})$ be a solution of the equation (A) *in a bounded domain G with smooth boundary ∂G. Then we obtain*

(4) $$|u(x)|\leqq \max_{\partial G}|u(x)|, \quad x\in G.$$

In case that $u(x)$ is not constant we have more precisely

(5) $$|u(x)|<\max_{\partial G}|u(x)|, \quad x\in G.$$

Corollary 3. *Under the same assumptions as in Corollary 2 except that* II*) *is replaced by the condition*

II#) *The function $f(x, u, 0)$ vanishes identically.*

we see that for any solution $u(x)\in C^2(G)\frown C^0(\overline{G})$ of (A)

6) See the footnote 5).

$$(6) \qquad \min_{\partial C} u(x) \leqq u(x) \leqq \max_{\partial G} u(x), \quad x \in G.$$

If, furthermore, $u(x)$ is not constant the following inequalities hold:

$$(7) \qquad \min_{\partial G} u(x) < u(x) < \max_{\partial G} u(x), \quad x \in G.$$

3. Applications. As an important application of the theorems stated in the preceding section we can prove the uniqueness of the solution of the Dirichlet problem for quasi-linear elliptic partial differential equations of the form

$$(\mathrm{B}) \qquad \sum_{i,j=1}^{n} a_{ij}(x, \operatorname{grad} u) \frac{\partial^2 u}{\partial x_i \partial x_j} = f(x, u, \operatorname{grad} u)$$

whose coefficients and free term are subjected to the following restrictions:

a) There exists a positive lower semi-continuous function $h(x, p)$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x, p) \xi_i \xi_j \geqq h(x, p) \| \xi \|^2$$

for every x and p under consideration and for every real vector ξ.

b) $| a_{ij}(x, p) - a_{ij}(x, p') | \leqq L_1 \| p - p' \| \quad (i, j = 1, 2, \cdots, n)$
with positive L_1.

c) The function $f(x, u, p)$ is non-decreasing with respect to u.

d) $| f(x, u, p) - f(x, u', p') | \leqq L_2 (| u - u' | + \| p - p' \|)$
with positive L_2.

By the Dirichlet problem $D(G; \varphi)$ for the equation (B) we mean the problem to seek a solution of (B) of class $C^2(\overline{G})$ satisfying the boundary condition: $u(x) = \varphi$ on ∂G, G being a bounded domain with smooth boundary. We assert that under these conditions the Dirichlet problem $D(G; \varphi)$ has at most one solution.

Indeed, let $u_0(x)$ be a fixed solution of the problem $D(G; \varphi)$ and $u(x)$ another solution. If we set $v(x) = u(x) - u_0(x)$, then $v(x)$ satisfies the quasi-linear elliptic equation

$$(\mathrm{B}') \qquad \sum_{i,j=1}^{n} A_{ij}(x, \operatorname{grad} v) \frac{\partial^2 v}{\partial x_i \partial x_j} = F(x, v, \operatorname{grad} v)$$

where

$$A_{ij}(x, \operatorname{grad} v) = a_{ij}(x, \operatorname{grad} v + \operatorname{grad} u_0)$$

and

$$F(x, v, \operatorname{grad} v) = f(x, v + u_0, \operatorname{grad} v + \operatorname{grad} u_0) - f(x, u_0, \operatorname{grad} u_0)$$
$$- \sum_{i,j=1}^{n} (a_{ij}(x, \operatorname{grad} v + \operatorname{grad} u_0) - a_{ij}(x, \operatorname{grad} u_0)) \frac{\partial^2 u_0}{\partial x_i \partial x_j}.$$

In view of the restrictions imposed on the equation (B) we can easily verify that Corollary 2 is applicable to the equation (B′) and we can therefore conclude that $v(x)$ vanishing on ∂G vanishes identically in G. Thus our assertion is demonstrated.

We shall next refer to the second application. Let us consider the equation (B) under the same assumptions as in the above uniqueness theorem. Suppose that the solutions $u(x)$ and $v(x)\in C^2(\overline{G})$ of (B) satisfy the inequality: $|u(x)-v(x)|\leqq\varepsilon$ on the boundary ∂G. Then the same inequality is valid in the whole of G. The proof of this fact is entirely similar to that of the uniqueness theorem. Let further a sequence $\{u_\nu(x)\}_{\nu=1}^{\infty}$ of solutions of (B) of class $C^2(\overline{G})$ be given. If the sequence converges uniformly on ∂G then it converges uniformly to a continuous function in the whole domain G. This is nothing but an extension of the Harnack's first theorem, though it is as yet impossible to examine whether the limit function is a solution of the original equation.

References

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