101. Open Basis and Continuous Mappings. II*)

By Sitiro HANAI
Osaka University of Liberal Arts and Education

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Let X and Y be T_1-spaces and let $f(X)=Y$ be a continuous mapping. f is said to be an S-mapping if the inverse image $f^{-1}(y)$ is separable[1] for each point y of Y. By the *open S-image*, we mean the image of an open continuous S-mapping. V. I. Ponomarev [4] has recently obtained the following theorem: *a T_1-space X has a point-countable open base if and only if X is an open S-image of a 0-dimensional metric space.*

In this note, we shall obtain an analogous theorem concernig the locally countable (star-countale) open base and we shall next investigate the open base of the inverse image space of an open continuous S-mapping.

1. We begin with proving the following theorem which is analogous to V. I. Ponomarev's theorem.

Theorem 1. *A T_1-space X has a locally countable (star-countable) open base if and only if X is an open S-image of a locally separable 0-dimensional metric space.*

Proof. As the "if" part is easily seen from our previous note ([1], Theorem 10, Remark 3), we shall prove the "only if" part. Since it is easily verified that X has a star-countable open base if and only if X has a locally countable open base, we deal with the case of the star-countable open base. Let X have a star-countable open base $\mathfrak{A}=\{U_\alpha\}$, then X is decomposed in such a way that $X=\bigcup_{\gamma\in\Gamma}A_\gamma$, $A_\gamma=\bigcup\{U_\alpha\in\mathfrak{A}_\gamma\}$, $A_\gamma\cap A_{\gamma'}=\phi$ for $\gamma\neq\gamma'$, $\gamma,\gamma'\in\Gamma$ where each $\mathfrak{A}_\gamma$ is a countable subfamily of $\mathfrak{A}$ [2, 6]. Then each A_γ has a countable open base $\mathfrak{A}_\gamma$ for each $\gamma\in\Gamma$. Let $\mathfrak{A}_\gamma=\{U_n^{(\gamma)}\mid n=1,2,\cdots\}$. For every point x of A_γ $\{U_n^{(\gamma)}\mid x\in U_n^{(\gamma)}, U_n^{(\gamma)}\in\mathfrak{A}_\gamma\}$ is countable. Let us denote this collection by $\{U_{n_i(x)}^{(\gamma)}\mid i=1,2,\cdots\}$, then, since X is a T_1-space, we have $\bigcap_{i=1}^{\infty}U_{n_i(x)}^{(\gamma)}=x$. If the intersection of all sets belonging to a countable subfamily $\{U_{n_i}^{(\gamma)}\}$ of $\mathfrak{A}_\gamma$ is a single point, then we define $\xi=(n_1,n_2,\cdots)$. Now let B_γ denote the set of all such ξ. We can define the topology

*) This note is a continuation of our previous note [1].

1) A set A is said to be separable when there exists a countable subset B of A such that $\bar{B}\supset A$. By the definition due to V. I. Ponomarev, S-mapping means the continuous mapping such that the inverse image $f^{-1}(y)$ is perfectly separable for each point y of Y, but we define here in the weaker sense than this.

of B_γ as a subspace of Baire's zero-space.[2] Then B_γ is a separable 0-dimensional metric space. If $x \in A_\gamma$ and $\bigcap_{i=1}^{\infty} U_{n_i}^{(\gamma)} = x$, we define a mapping f_γ of B_γ onto A_γ by $f_\gamma(\xi)=x$ where $\xi=(n_1, n_2, \cdots)$. To prove that f_γ is an open continuous mapping, it is sufficient to show that f_γ transforms the base for the neighborhood system of ξ to that of $f_\gamma(\xi)$ for any point ξ of B_γ. Let $V_n(\xi)=\left\{\xi' \mid \rho(\xi, \xi')<\frac{1}{n}, \xi' \in B_\gamma\right\}$, then $\{V_n(\xi) \mid n=1, 2, \cdots\}$ is a base for the neighborhood system of ξ. Let $f_\gamma(\xi)=x$ and let $U(x)$ be any neighborhood of x where $\xi=(n_1, n_2 \cdots)$, then, since $\bigcap_{i=1}^{\infty} U_{n_i}^{(\gamma)}=x$, we can find n_k such that $U_{n_k}^{(\gamma)} \subset U(x)$. Then $f_\gamma(V_{k+1}(\xi)) \subset U_{n_k}^{(\gamma)} \subset U(x)$. Therefore f_γ is an open continuous mapping of B_γ onto A_γ. Moreover, since B_γ is separable, the inverse image $f_\gamma^{-1}(x)$ is separable for every point x of A_γ. Hence f_γ is an open continuous S-mapping. For each $\gamma \in \Gamma$, let C_γ is a topological space such that C_γ is homeomorphic to B_γ and let $C_\gamma \cap C_{\gamma'}=\phi$ for $\gamma \neq \gamma'$. We define the topology of $T=\bigcup_{\gamma \in \Gamma} C_\gamma$ as follows: for each point t of T such that $t \in C_\gamma$, the base for the open neighborhood system of t is that of t of the space C_γ. Then T is a locally separable 0-dimensional metric space. Let φ_γ be the above homeomorphism between C_γ and B_γ. We define a mapping f of T onto X as follows: if $t \in C_\gamma$, then $f(t)=f_\gamma \varphi_\gamma(t)$. Then it is easy to see that f is an open continuous S-mapping of T onto X. This completes the proof.

As an immediate consequence of Theorem 1, we get the following corollary.

Corollary 1. *A T_1-space X is perfectly separable if and only if X is an open S-image of a separable 0-dimensional metric space.*

2. In this section, we deal with the open basis of inverse image spaces of open continuous S-mappings.

Theorem 2. *Let X be a topological space and let Y be a topological space with a locally countable (star-countable) open base. If $f(X)=Y$ is an open continuous S-mapping, then X has a locally countable (star-countable) open base if and only if X has a point-countable open base.*

Proof. As the "only if" part is obvious, we shall prove the "if" part. By the same argument as that of Theorem 1, we can decompose Y in such a way that $Y=\bigcup_{\gamma \in \Gamma} A_\gamma$ where $A_\gamma \cap A_{\gamma'}=\phi$ for $\gamma \neq \gamma'$

2) Let B be the set of all points x such that $x=(n_1, n_2, n_3, \cdots)$ where each n_i is a positive integer. Let $x=(n_1, n_2, n_3, \cdots)$ and $y=(m_1, m_2, m_3, \cdots)$ be any points of B. If $n_i=m_i$ for $i<k$ and $n_k \neq m_k$, then we define the metric $\rho(x, y)=\frac{1}{k}$. When we define $\rho(x, y)$ for any two points x and y of B, B is said to be Baire's zero-space.

and each A_γ is perfectly separable open subspace of Y. Since f is an open continuous S-mapping, each $f^{-1}(A_\gamma)$ is separable ([8], Lemma 2). Then, since X has a point-countable open base, each $f^{-1}(A_\gamma)$ is perfectly separable. Since $X=\bigcup_{\gamma\in\Gamma} f^{-1}(A_\gamma)$ and $f^{-1}(A_\gamma)\cap f^{-1}(A_{\gamma'})=\phi$ for $\gamma\neq\gamma'$, X has a star-countable open base. This completes the proof.

Corollary 2. *A topological space X has a locally countable (star-countable) open base if and only if the product space $X\times Y$ has a locally countable (star-countable) open base for any topological space Y with a locally countable (star-countable) open base.*

Proof. As the "if" part is obvious, we need only prove the "only if" part. Since X has a star-countable open base, X is decomposed in such a way that $X=\bigcup_{\gamma\in\Gamma} A_\gamma$ and $A_\gamma\cap A_{\gamma'}=\phi$ for $\gamma\neq\gamma'$ and each A_γ is a perfectly separable open subspace of X. Then $X\times Y=(\bigcup_{\gamma\in\Gamma} A_\gamma)\times Y=\bigcup_{\gamma\in\Gamma}(A_\gamma\times Y)$ and $(A_\gamma\times Y)\cap(A_{\gamma'}\times Y)=\phi$ for $\gamma\neq\gamma'$. Let f_γ be the projection of $A_\gamma\times Y$ onto Y, then it is easy to see that f_γ is an open continuous S-mapping. Then, by virtue of Theorem 2, $A_\gamma\times Y$ has a star-countable open base. Therefore $X\times Y$ has a star-countable open base. This completes the proof.

Theorem 3. *Let X be a regular T_1-space and let Y be a locally separable metric space. If $f(X)=Y$ is an open continuous S-mapping, then X is a locally separable metric space if and only if X has a point-countable open base.*

Proof. As the "only if" part is obvious, we need only prove the "if" part. Since Y is a locally separable metric space, Y has a star-countable open base. Then, by virtue of Theorem 2, X has a star-countable open base. Therefore X is locally separable and locally metrizable. Since X has a star-countable open base, X is strongly paracompact. Hence X is metrizable by Nagata-Smirnov's theorem [3, 5]. This completes the proof.

As an immediate result of Theorem 3, we get the following theorem which includes the well-known theorem due to A. H. Stone ([8], Theorem 4).

Theorem 4. *Let X be a regular T_1-space with a point-countable open base and let Y be a regular T_1-space. If $f(X)=Y$ is an open continuous S-mapping, then Y is a locally separable metric space if and only if X is a locally separable metric space.*

In conclusion, we shall give an example which shows that we can not drop the assumption that X has a point-countable open base in Theorems 2, 3, and 4.

Example. Let $X=[0,1]$ that is, the closed interval of the real line. We define the topology of X as follows: if $x\neq 1$ and $x\in X$, then the collection of all semi-open intervals of the form $[x, y)$ with $x<y\leqq 1$ is the base for the neighborhood system of x and if $x=1$, then the single point x is itself open (cf. [7]). It is easy to see that X is a separable normal T_1-space but not perfectly separable. Hence X is not metrizable. Therefore X has no point-countable open base. Now let $Y=[0, 1]$ be the subspace of the real line, then Y is a separable metric space and hence Y has a star-countable open base. Let f be the projection of $X\times Y$ onto Y, then f is an open continuons S-mapping. On the other hand, it is easy to see that $X\times Y$ has no point-countable open base. In fact, suppose on the contrary that $X\times Y$ has a point-countable open base. Then, since $X\times Y$ is separable, $X\times Y$ is perfectly separable. Hence X is perfectly separable. This contradicts the fact that X is not perfectly separable.

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