158. On Distributions and Spaces of Sequences. III

On Powers of the Distributions

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1. In the previous articles [1] and [2] published under the same title, we considered the convergent or divergent generalized distributions. In this article we investigate the topologies defined by the powers of sequences and the derived generalized distributions introduced by these topologies. We also show here corrections to the errors in the previous articles [1], [2], and [3].

2. **Definition 1.** We denote σ_α the topology induced in the space $\mathfrak{D}$ by neighbourhoods $U_n(\varphi)$ of φ:

$U_n(\varphi)=\{\psi\,|\,e^{ia(\psi)}|\psi|^\alpha-e^{ia(\varphi)}|\varphi|^\alpha\in W_{\mathfrak{D}'}(0)\}$ where $a(\varphi), a(\psi)$ means the argument of the complex valued function φ or ψ and $\alpha>0$.

Then we see the following

Lemma 1. *The topology σ_α defines an uniform Hausdorff structure of the space $\mathfrak{D}$.*

Proof. It is easily seen that σ_α satisfies the conditions of Hausdorff space, and also the conditions (U_I), (U_{II}) of the uniformity [4]. Since for any $W_{\mathfrak{D}'}(0)$ there exists $W'_{\mathfrak{D}'}(0)$ such that $W'-W'\subset W$, the condition (U_{III}) is satisfied.

Definition 2. We say that $\{\varphi_n\}$ is a *Cauchy sequence in σ_α*, if and only if the following condition is satisfied:

For any $W_{\mathfrak{D}'}(0)$, there exists an integer N such that $\varphi_n\in U_W(\varphi_n)$ for all $m, n>N$.

Definition 3. Suppose that two Cauchy sequences $\{\varphi_n\}$ and $\{\psi_n\}$ satisfy the following condition: for any $W_{\mathfrak{D}'}(0)$, there exists an integer N such that $\varphi_n\in U_W(\psi_m)$, $\psi_m\in U_W(\varphi_n)$ for all $m, n>N$. Then we say that $\{\varphi_n\}$ is *equivalent* to $\{\psi_m\}$.

Lemma 2. *The topology σ_α is not compatible with linear operation.*

Proof. We show here that there exist two sequences $\{\varphi_j\}$ and $\{\psi_j\}$ such that though both φ_j and ψ_j converge to 0, the sequence $\varphi_j+\psi_j$ does not converge to 0.

Example 1. φ_n is defined by $\varphi_n^0*\rho_{n^2}$ where

$$\varphi_n^0=\begin{cases} n^{1/\alpha} & \text{for} \quad 1/n\leq x\leq 2/n \\ -n^{1/\alpha} & \text{for} \quad -2/n\leq x\leq -1/n\end{cases}$$

and ρ_{n^2} is a mollifier defined by L. Schwartz [5] with a compact support of length $1/n^2$.

ψ_n is defined by $\psi_n=\psi_n^0*\rho_{n^2}$ where

$$\psi_n^0=\begin{cases} n^{1/\alpha} & \text{for} \quad 1/n\leq x\leq 2/n \\ -n^{1/\alpha} & \text{for} \quad -3/n\leq x\leq -2/n. \end{cases}$$

$(\varphi_n+\psi_n)$ converges to $(2^\alpha-2)\cdot\delta$ in σ_α topology. For $\alpha\neq 1$ the limit is not 0.

Since Lemma 2 and this example hold, we see, of course, that σ_α is not locally convex topology.

The proof of convexity in [3] is similarly not right. We want to delete Lemma 3 in p. 341 and p. 342 in [3].

Now we obtain the following

Thorem 1. *The space $\mathfrak{D}$ with topology σ_α is Hausdorff uniform space, but neither locally convex nor linear.*

Lemma 3. *$\{(\varphi_j^T)^{1/\alpha}\}$ makes a Cauchy sequence (in σ_α) for any $T\in\mathfrak{D}'$ and any $\{\varphi_j^T\}$. They are equivalent to each other and belong to the same class. The sequences $\{\varphi_j^T\}$ themselves, however, are not always equivalent (in σ_α).*

Proof. The former part of the property is evident. The latter part of the proposition is proved by the following examples $\{\varphi_j\}$, $\{\psi_j\}$ and $\{\chi_j\}$ for $\alpha=2$:

Example 2. $\varphi_j=\bar{\varphi}_j*\rho_{j^2}$, $\psi_j=\bar{\psi}_j*\rho_{j^2}$, $\chi_j=\bar{\chi}_j*\rho_{j^2}$, where $\varepsilon_j\downarrow 0$ and

$$\bar{\varphi}_j\equiv\begin{cases} 0, & \text{for} \quad |x|>\varepsilon_j/2 \\ 1/\sqrt{\varepsilon_j} & \text{for} \quad |x|<\varepsilon_j/2, \end{cases} \qquad \bar{\psi}_j\equiv\begin{cases} 0 & \text{for} \quad |x|>\varepsilon_j/2 \\ 1/\sqrt[3]{\varepsilon_j} & \text{for} \quad |x|<\varepsilon_j/2 \end{cases}$$

$$\bar{\chi}_j\equiv\begin{cases} 0, & \text{for} \quad |x|>\varepsilon_j/2 \\ 1/\sqrt[3]{\varepsilon_j^2} & \text{for} \quad |x|<\varepsilon_j/2. \end{cases}$$

$(\varphi_j)^{1/2}$, $(\psi_j)^{1/2}$, and $(\chi_j)^{1/2}$ converge to 0 in σ_α, but φ_j converges to $\sqrt{\delta}$, ψ_j converges to 0, χ_j diverges in σ_α.

Definition 4. We donote the equivalent sequence $\{(\varphi_j^T)^{1/\alpha}\}$ by $T^{1/\alpha}$.

Theorem 2. *$\{(\varphi_j^T)^{1/\alpha}\}$ defines the generalized distribution $T^{1/\alpha}$,* for any $T\in D'$ and for any $\{\varphi_j^T\}$.

3. We show here the properties of the topologies defined by the linear shift of neighbourhood system of 0, and correct the statement about Example 2, p. 330 in [1].

Definition 5. *We denote τ_α the topology which is defined by the following set of neighbourhood $U_W(\psi)$ of ψ such that for $\psi=0$.*

$U_W(0)=\{\varphi\,|\,e^{i\alpha(\varphi)}|\varphi|^\alpha\in W_{D'}(0)\}$ where $W_{\mathfrak{D}'}(0)$ is a neighbourhood of 0 in topology $\tau_{\mathfrak{D}'}$, and for $\psi\neq 0: U_W(\psi)=\{\varphi\,|\,\varphi-\psi\in U_W(0)\}$.

Lemma 4. *τ_{α_1} is neither stronger than τ_{α_2} nor weaker than τ_{α_2} for $\alpha_1\neq\alpha_2$.*

Proof. In Example 3 we construct the sequence $\{\varphi_n\}$ which has properties such that φ_n converge to 0 in the sense of topology τ_{α_1}

and converge to a function $\varphi(x)\equiv c$, $c\neq 0$ in topology τ_{α_2} for $\alpha_1\neq\alpha_2$.

Example 3. $\varphi_n=\bar{\varphi}_n*\rho_{n^2}$,

$$\text{where } \bar{\varphi}_n=\begin{cases} 1 & \text{for } \dfrac{k}{n}<x\leqq\dfrac{k}{n}+\dfrac{1}{3n} \\ -1/2^{1/\alpha_1} & \text{for } \dfrac{k}{n}+\dfrac{1}{3n}<x\leqq\dfrac{k+1}{n} \end{cases} \quad (k \text{ is an integer } 0\leqq k<n-1).$$

Remark. *The sequence* $\{\psi_n\}$ *shown in* Example 4 *has the properties such that* ψ_n *converges to* 0 *in the sense of topology* τ_{α_1} *and diverges in* τ_{α_2} *for* $\alpha_1<\alpha_2$.

Example 4. $\psi_n=\bar{\psi}_n*\rho_{n^2}$, $\varepsilon_n\downarrow 0$

$$\bar{\psi}_n=\begin{cases} 0 & \text{for } |x|>\varepsilon_n/2 \\ 1/\varepsilon_n^{1/(\alpha_1+\varepsilon)} & \text{for } |x|<\varepsilon_n/2, \quad \text{where } \varepsilon=\dfrac{\alpha_2-\alpha_1}{2}. \end{cases}$$

Lemma 5. *For any positive valued function* $\varphi(x)\in\mathfrak{D}$, *there exists a sequence* $\{\varphi_n^0\}$ *such that* φ_n^0 *converges to* 0 *in* τ_{α_1} *and* φ_n^0 *converges to* φ *in* τ_{α_2} *for* $\alpha_1<\alpha_2$.

Proof. Without loss of generality we can assume that the function $\varphi(x)$ has carrier in the interval $[0, 1]$.

Let $\varphi_n^0(x)$ be the function:

$\varphi_n^0(x)=\sum_j\rho_{n^n}*\bar{\varphi}_{nj}^0(x)$ where ρ_{n^n} means mollifier of order n^n, and for $j=3m$, $0\leq j<n-3$ for positive integer m.

$$\bar{\varphi}_{nj}^0(x)=\begin{cases} a_i+b_i; & \dfrac{i}{n}+\dfrac{j}{n^2}\leq x<\dfrac{i}{n}+\dfrac{j+1}{n^2} \\ -a_i\,; & \dfrac{i}{n}+\dfrac{j+1}{n^2}\leq x<\dfrac{i}{n}+\dfrac{j+3}{n^2} \end{cases}$$

where a_i, b_i satisfy the relations $(a_i+b_i)^{\alpha_1}-2a_i^{\alpha_1}=0$, $(a_i+b_i)^{\alpha_2}-2a_i^{\alpha_2}=\varphi(i/n)$ i.e. $a_i=\{\varphi(i/n)/(2^{\alpha_2/\alpha_1}-2)\}^{1/\alpha_2}$ and $b_i=a_i(2^{1/\alpha_1}-1)$. We see that $\varphi_n^0(x)$ converges to 0 in τ_{α_1}, and converges to $\varphi(x)$ in τ_{α_2}.

Lemma 6. The topology τ_α is not compatible with linear operation in the space $\mathfrak{D}$.

Proof. The example in the proof of Lemma 2 shows also that this lemma holds.

From Lemmas 1, 2, 4, and Definition 4 we obtain the following

Theorem 3. *The space* $(\mathfrak{D}, \tau_\alpha)$ *is Hausdorff uniform space, but is neither locally convex nor linear. Two topological spaces with different* α_1 *and* α_2 *cross with each other, namely each space is not inclued in the other space and the one topology is neither stronger nor weaker than the other.*

In this space $(\mathfrak{D}, \sigma_\alpha)$ *we obtained* $T^{1/\alpha}$ *for any* $T\in\mathfrak{D}'$ *and for any* $\{\varphi_j^T\}$. *In the space* $(\mathfrak{D}, \tau_\alpha)$, *however, it does not necessarily hold.*

We show here the latter part of the Theorem by the following Example 5.

Example 5. Let φ_j be the function such that $\varphi_j=\bar{\varphi}_j*\rho_{j^j}$ where

$$\overline{\varphi}=\begin{cases}0 & \text{for} \quad |x|>(1/2j)\\ j^{1/\alpha} & \text{for} \quad |x|<(1/2j).\end{cases}$$

Then we see from the following equations that $\{\varphi_j\}$ does not make Cauchy sequence but $\{\varphi_{j^2}\}$ makes Cauchy sequence.

$$\int(\overline{\varphi}_n-\overline{\varphi}_m)\,dx=(n^{1/\alpha}-m^{1/\alpha})^\alpha\times(1/n)-m\{(1/m)-(1/n)\}$$
$$=(1-(m/n)^{1/\alpha})^\alpha-(1-m/n), \quad \text{for} \quad n>m.$$

Hence if $n=0(m)$, for example $n=2m$, then

$$\lim_{m\to\infty}\int(\varphi_n-\varphi_m)\,dx=\lim_{m\to\infty}\{(1-2^{-1/\alpha})^\alpha-(1-2^{-1})\}\neq 0$$

and if $n=0(m^2)$, then $\lim_{m\to\infty}\int(\varphi_n-\varphi_m)\,dx=0$.

References

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