15. *On Absolute Summability Factors of Infinite Series*

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1. *Definitions and Notations.* Let s_n denote the n-th partial sum of a given infinite series $\sum a_n$. We write

$$t_n = \frac{1}{L_n} \sum_{\nu=1}^{n} \frac{1}{\nu} s_\nu,$$

where

$$L_n = \sum_{\nu=1}^{n} \frac{1}{\nu} \backsim \log n, \quad \text{as } n \to \infty.$$

We say that the series $\sum a_n$ is absolutely summable $\left(R, \frac{1}{n}\right)$, or summable $\left|R, \frac{1}{n}\right|$, if the sequence $\{t_n\}$ is of bounded variation,[1] that is, the series $\sum |t_n - t_{n+1}|$ is convergent. It may be observed that this method of summability is equivalent to the absolute summability method defined by means of the auxiliary sequence

$$\frac{1}{\log n} \sum_{\nu=1}^{n} \frac{1}{\nu} s_\nu \text{ [2]}$$

known as the Riesz logarithmic mean of $\{s_n\}$.[3]

A sequence $\{\lambda_n\}$ is said to be convex[4] if

$$\Delta^2 \lambda_n = \Delta^2(\lambda_n) \geq 0, \quad n = 1, 2, \cdots,$$

where

$$\Delta^2(\lambda_n) = \Delta(\Delta \lambda_n) = \Delta \lambda_n - \Delta \lambda_{n+1}$$

and

$$\Delta \lambda_n = \Delta(\lambda_n) = \lambda_n - \lambda_{n+1}.$$

Let $\{\lambda_n\}$ be a monotonic increasing sequence such that

$$\lambda_n \to \infty, \quad \text{as } n \to \infty.$$

We write

$$A_\lambda(\omega) = A_\lambda^0(\omega) = \sum_{\lambda_n \leq \omega} a_n,$$

and, for $r > 0$,

$$A_\lambda^r(\omega) = \sum_{\lambda_n \leq \omega} (\omega - \lambda_n)^r a_n = r \int_0^\omega (\omega - \tau)^{r-1} A_\lambda(\tau) d\tau.$$

For $r \geq 0$, we write

$$R_\lambda^r(\omega) = A_\lambda^r(\omega) / \omega^r.$$

$\sum a_n$ is said to be absolutely summable (R, λ_n, r), or summable

1) Symbolically $\{t_n\} \in BV$.

2) This can be easily seen by virtue of Lemma 3 of Iyer's paper [4], which states that the sequence $\{\omega_n\} \equiv \left\{\left(1 + \frac{1}{2} + \cdots + \frac{1}{n+1}\right) \Big/ \log n\right\}$ is of bounded variation, when we note that ω_n is strictly positive for $n \geq 2$.

3) Hardy [3], § 4.16.

4) Zygmund [8], p. 58.

$|R, \lambda_n, r|$, $r \geq 0$, if $R_\lambda^r(\omega)$ is a function (of ω) of bounded variation over the infinite interval (k, ∞), where k is some finite positive number.[5]

It has been pointed out by Prof. Bosanquet that summability $|R, \log n, 1|$ is equivalent to summability $\left|R, \frac{1}{n}\right|$.[6]

Writing

$$t_n = \frac{1}{\Lambda_n} \sum_{\nu=1}^{n} \mu_\nu s_\nu,$$

where $\mu_n > 0$ for all n, and $\Lambda_n = \sum_{\nu=1}^{n} \mu_\nu \to \infty$,

we shall say that the series $\sum a_n$ is absolutely summable (R, μ_n), or summable $|R, \mu_n|$, if $\{t_n\} \in BV$.

2. *Introduction.* The following result is known.

Theorem A.[7] If $\{\lambda_n\}$ is a convex sequence such that the series $\sum n^{-1}\lambda_n$ is convergent and the sequence $\{s_n\}$ is bounded, then the series $\sum a_n \lambda_n \log n$ is summable $\left|R, \frac{1}{n}\right|$.[8]

It may be remarked that Theorem A was used for proving a certain result on the localization of summability $|R, \log n, 1|$ of a Lebesgue-Fourier series with factors.

The object of the present paper is to demonstrate an extension of Theorem A.

3.1. We establish the following theorem.

Theorem. *If* $s_n = a_1 + \cdots + a_n$, *and*

$$\sum_{n=2}^{\infty} |s_n - a_1||\lambda_n||\Delta\varphi_n| < \infty \tag{3.1.1}$$

$$\sum_{n=2}^{\infty} |s_n - a_1||\lambda_n||\varphi_n| \frac{\mu_n}{\Lambda_n} < \infty \tag{3.1.2}$$

and

$$\sum_{n=2}^{\infty} |s_n - a_1||\varphi_{n+1}||\Delta\lambda_n| < \infty, \tag{3.1.3}$$

then the series $\sum_{n=1}^{\infty} a_n \lambda_n \varphi_n$ *is summable* $|R, \mu_n|$.

3.2. *Proof of the Theorem.*

Writing
$$T_n = \sum_{\nu=1}^{n} C_\nu,$$
$$C_n = a_n \lambda_n \varphi_n$$

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5) Obrechkoff [5], [6].

6) Bosanquet [2].

7) Bhatt [1].

8) Bhatt states in his enunciation $|R, \log n, 1|$ in place of absolute Riesz logarithmic summability on account of the equivalence of these two methods and the fact that the latter is equivalent to the method $\left|R, \frac{1}{n}\right|$.

and $$R_n=\frac{1}{\Lambda_n}\sum_{\nu=1}^{n}\mu_\nu T_\nu,$$

we have

$$\begin{aligned}R_n-R_{n+1}&=\frac{1}{\Lambda_n}\sum_{\nu=1}^{n}\mu_\nu T_\nu-\frac{1}{\Lambda_{n+1}}\sum_{\nu=1}^{n+1}\mu_\nu T_\nu\\
&=\frac{\mu_{n+1}}{\Lambda_n\Lambda_{n+1}}\sum_{\nu=1}^{n}\mu_\nu T_\nu-\frac{\mu_{n+1}T_{n+1}}{\Lambda_{n+1}}\\
&=\frac{\mu_{n+1}}{\Lambda_n\Lambda_{n+1}}\sum_{\nu=1}^{n-1}\Lambda_\nu\Delta T_\nu+\frac{\mu_{n+1}}{\Lambda_{n+1}}(T_n-T_{n+1})\\
&=-\frac{\mu_{n+1}}{\Lambda_n\Lambda_{n+1}}\sum_{\nu=1}^{n-1}\Lambda_\nu C_{\nu+1}-\frac{\mu_{n+1}}{\Lambda_{n+1}}C_{n+1}\\
&=-\frac{\mu_{n+1}}{\Lambda_n\Lambda_{n+1}}\sum_{\nu=1}^{n}\Lambda_\nu C_{\nu+1}\\
&=-\frac{\mu_{n+1}}{\Lambda_n\Lambda_{n+1}}\sum_{\nu=1}^{n}a_{\nu+1}\lambda_{\nu+1}\varphi_{\nu+1}\Lambda_\nu\\
&=-\frac{\mu_{n+1}}{\Lambda_n\Lambda_{n+1}}\Big[\sum_{\nu=1}^{n-1}(s_{\nu+1}-a_1)\Delta(\lambda_{\nu+1}\varphi_{\nu+1}\Lambda_\nu)\\
&\qquad+(s_{n+1}-a_1)\lambda_{n+1}\varphi_{n+1}\Lambda_n\Big]\\
&=-\frac{\mu_{n+1}}{\Lambda_n\Lambda_{n+1}}\Big[\sum_{\nu=1}^{n-1}(s_{\nu+1}-a_1)\{\lambda_{\nu+1}\Lambda_{\nu+1}\Delta\varphi_{\nu+1}\\
&\qquad-\lambda_{\nu+1}\varphi_{\nu+1}\mu_{\nu+1}+\varphi_{\nu+2}\Lambda_{\nu+1}\Delta\lambda_{\nu+1}\}\\
&\qquad+(s_{n+1}-a_1)\lambda_{n+1}\varphi_{n+1}\Lambda_n\Big].\end{aligned}$$

Hence

$$\begin{aligned}\sum_{n=2}^{\infty}|R_n-R_{n+1}|&\leq\sum_{n=2}^{\infty}\left|\frac{\mu_{n+1}}{\Lambda_n\Lambda_{n+1}}\sum_{\nu=1}^{n-1}(s_{\nu+1}-a_1)\lambda_{\nu+1}\Lambda_{\nu+1}\Delta\varphi_{\nu+1}\right|\\
&\quad+\sum_{n=2}^{\infty}\left|\frac{\mu_{n+1}}{\Lambda_n\Lambda_{n+1}}\sum_{\nu=1}^{n-1}(s_{\nu+1}-a_1)\lambda_{\nu+1}\varphi_{\nu+1}\mu_{\nu+1}\right|\\
&\quad+\sum_{n=2}^{\infty}\left|\frac{\mu_{n+1}}{\Lambda_n\Lambda_{n+1}}\sum_{\nu=1}^{n-1}(s_{\nu+1}-a_1)\varphi_{\nu+2}\Lambda_{\nu+1}\Delta\lambda_{\nu+1}\right|\\
&\quad+\sum_{n=2}^{\infty}\left|\frac{\mu_{n+1}}{\Lambda_{n+1}}(s_{n+1}-a_1)\lambda_{n+1}\varphi_{n+1}\right|\\
&=\Sigma_1+\Sigma_2+\Sigma_3+\Sigma_4,\quad\text{say.}\end{aligned}$$

$$\begin{aligned}\Sigma_1&\leq\sum_{n=2}^{\infty}\frac{\mu_{n+1}}{\Lambda_n\Lambda_{n+1}}\sum_{\nu=1}^{n-1}|(s_{\nu+1}-a_1)\lambda_{\nu+1}\Lambda_{\nu+1}\Delta\varphi_{\nu+1}|\\
&=\sum_{n=1}^{\infty}\frac{\mu_{n+2}}{\Lambda_{n+1}\Lambda_{n+2}}\sum_{\nu=1}^{n}|(s_{\nu+1}-a_1)\lambda_{\nu+1}\Lambda_{\nu+1}\Delta\varphi_{\nu+1}|\\
&=\sum_{\nu=1}^{\infty}|(s_{\nu+1}-a_1)\lambda_{\nu+1}\Lambda_{\nu+1}\Delta\varphi_{\nu+1}|\sum_{n=\nu}^{\infty}\frac{\mu_{n+2}}{\Lambda_{n+1}\Lambda_{n+2}}\\
&=\sum_{\nu=1}^{\infty}|(s_{\nu+1}-a_1)\lambda_{\nu+1}\Lambda_{\nu+1}\Delta\varphi_{\nu+1}|\frac{1}{\Lambda_{\nu+1}},\end{aligned}$$

(since $\sum_{\nu}^{\infty}\frac{\mu_{n+2}}{\Lambda_{n+1}\Lambda_{n+2}}=\frac{1}{\Lambda_{\nu+1}}$, as $\Lambda_n\to\infty$ with n),

$$= \sum_{\nu=2}^{\infty} |s_\nu - a_1| |\lambda_\nu| |\Delta\varphi_\nu|$$

(3.2.1) $$< \infty,$$

by (3.1.1).

$$\sum_2 \leq \sum_{n=2}^{\infty} \frac{\mu_{n+1}}{\Lambda_n \Lambda_{n+1}} \sum_{\nu=1}^{n-1} |(s_{\nu+1} - a_1)\lambda_{\nu+1}\varphi_{\nu+1}\mu_{\nu+1}|$$

$$= \sum_{n=1}^{\infty} \frac{\mu_{n+2}}{\Lambda_{n+1}\Lambda_{n+2}} \sum_{\nu=1}^{n} |(s_{\nu+1} - a_1)\lambda_{\nu+1}\varphi_{\nu+1}\mu_{\nu+1}|$$

$$= \sum_{\nu=1}^{\infty} |(s_{\nu+1} - a_1)\lambda_{\nu+1}\varphi_{\nu+1}\mu_{\nu+1}| \sum_{n=\nu}^{\infty} \frac{\mu_{n+2}}{\Lambda_{n+1}\Lambda_{n+2}}$$

$$= \sum_{\nu=1}^{\infty} |(s_{\nu+1} - a_1)\lambda_{\nu+1}\varphi_{\nu+1}\mu_{\nu+1}| \frac{1}{\Lambda_{\nu+1}}$$

$$= \sum_{\nu=2}^{\infty} |s_\nu - a_1| |\lambda_\nu| |\varphi_\nu| \frac{\mu_\nu}{\Lambda_\nu}$$

(3.2.2) $$< \infty,$$

by (3.1.2).

$$\sum_3 \leq \sum_{n=2}^{\infty} \frac{\mu_{n+1}}{\Lambda_n \Lambda_{n+1}} \sum_{\nu=1}^{n-1} |(s_{\nu+1} - a_1)\varphi_{\nu+2}\Lambda_{\nu+1}\Delta\lambda_{\nu+1}|$$

$$= \sum_{n=1}^{\infty} \frac{\mu_{n+2}}{\Lambda_{n+1}\Lambda_{n+2}} \sum_{\nu=1}^{n} |(s_{\nu+1} - a_1)\varphi_{\nu+2}\Lambda_{\nu+1}\Delta\lambda_{\nu+1}|$$

$$= \sum_{\nu=1}^{\infty} |(s_{\nu+1} - a_1)\varphi_{\nu+2}\Lambda_{\nu+1}\Delta\lambda_{\nu+1}| \sum_{n=\nu}^{\infty} \frac{\mu_{n+2}}{\Lambda_{n+1}\Lambda_{n+2}}$$

$$= \sum_{\nu=1}^{\infty} |(s_{\nu+1} - a_1)\varphi_{\nu+2}\Lambda_{\nu+1}\Delta\lambda_{\nu+1}| \frac{1}{\Lambda_{\nu+1}}$$

$$= \sum_{\nu=2}^{\infty} |s_\nu - a_1| |\varphi_{\nu+1}| |\Delta\lambda_\nu|$$

(3.2.3) $$< \infty,$$

by (3.1.3).

Lastly,

$$\sum_4 = \sum_{n=3}^{\infty} |s_n - a_1| |\lambda_n| |\varphi_n| \frac{\mu_n}{\Lambda_n}$$

(3.2.4) $$< \infty,$$

by (3.1.2).

Thus, collecting the inequalities (3.2.1), (3.2.2), (3.2.3), and (3.2.4), we have

$$\sum_{n=1}^{\infty} |R_n - R_{n+1}| < \infty,$$

that is, $\sum_{n=1}^{\infty} a_n \lambda_n \varphi_n$ is summable $|R, \mu_n|$.

This completes the proof of our theorem.

We give here a direct corollary of our theorem, which is somewhat more general than Theorem A.

COROLLARY. *If* $\{\lambda_n\}$ *is monotonic non-increasing, that is,* $\Delta\lambda_n \geq 0$, *and* $\sum n^{-1}\lambda_n$ *is convergent, and* $\{s_n\}$ *is bounded, then* $\sum a_n \lambda_n \log n$ *is* *summable* $\left|R, \frac{1}{n}\right|$.

To prove this we need the following lemma, suggested by Dr. Pati, which is more general in form than Lemma 3 of Pati [7].

LEMMA. *If* $\{\lambda_n\}$ *is monotonic non-increasing, and* $\sum n^{-1}\lambda_n$ *is convergent, then* $\sum_1^\infty \log(n+1)\Delta\lambda_n<\infty$.

Proof. First, we show that if $\Delta\lambda_n\geq 0$ and $\sum n^{-1}\lambda_n<\infty$, then

$$\lambda_n \log n=O(1), \quad \text{as } n\to\infty.$$

Now, since λ_n is monotonic non-increasing, we have

$$\lambda_m \log m=O\left\{\lambda_m\left(\sum_1^m n^{-1}\right)\right\}=O\left(\sum_1^m n^{-1}\lambda_n\right)=O(1),$$

as $m\to\infty$.

Now following Pati,[9] we have

$$\sum_1^m \log(n+1)\Delta\lambda_n=\lambda_1 \log 2-\sum_1^{m-1}\Delta\{\log(n+1)\}\lambda_{n+1}-\lambda_{m+1}\log(m+1)$$
$$=O(1),$$

since

$$\Delta\{\log(n+1)\}=\log(n+1)-\log(n+2)=O\{1/(n+1)\},$$

and

$$\lambda_n \log n=O(1),$$

as proved above.

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9) Pati [7], p. 276, Lemma 3.