41. *Extension of a Theorem of Hyslop on Absolute Cesàro Summability*

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1. *Definitions and notations.* We shall denote the nth Cesàro-sum, Cesàro-mean and Cesàro-transformed term of order κ $(\kappa>-1)$ of the series $\sum a_n$ by S_n^κ, s_n^κ and a_n^κ respectively, and the corresponding sum, mean and term for the series $\sum \lambda_n a_n$ by $S_{n,\lambda}^\kappa$, $s_{n,\lambda}^\kappa$ and $a_{n,\lambda}^\kappa$ respectively.

Thus
$$s_n^\kappa=\frac{S_n^\kappa}{A_n^\kappa}=\frac{1}{A_n^\kappa}\sum_{\nu=0}^{n}A_{n-\nu}^{\kappa-1}s_\nu=\frac{1}{A_n^\kappa}\sum_{\nu=0}^{n}A_{n-\nu}^{\kappa}a_\nu,$$
where A_n^κ is defined by the identity
$$(1-x)^{-\kappa-1}=\sum A_n^\kappa x^n \qquad (|x|<1),$$
and
$$a_n^\kappa=s_n^\kappa-s_{n-1}^\kappa.$$
Similarly,
$$s_{n,\lambda}^\kappa=\frac{S_{n,\lambda}^\kappa}{A_n^\kappa}=\frac{1}{A_n^\kappa}\sum_{\nu=0}^{n}A_{n-\nu}^{\kappa}\lambda_\nu a_\nu$$
and
$$a_{n,\lambda}^\kappa=s_{n,\lambda}^\kappa-s_{n-1,\lambda}^\kappa.$$

A series is said to be absolutely summable (C,κ), or summable $|C,\kappa|$, $\kappa>-1$, if
$$\sum|a_n^\kappa|=\sum|s_n^\kappa-s_{n-1}^\kappa|<\infty.$$

We observe that, by a well-known identity, due to Kogbetliantz,*)
$$a_n^\kappa=s_n^\kappa-s_{n-1}^\kappa=n^{-1}t_n^\kappa=n^{-1}(A_n^\kappa)^{-1}T_n^\kappa,$$
where t_n^κ and T_n^κ are the nth Cesàro-mean and sum of order κ of the sequence $\{na_n\}$. Thus the summability $|C,\kappa|$ of $\sum a_n$ is the same as the convergence of the series $\sum n^{-1}|t_n^\kappa|$, or $\sum n^{-1}(A_n^\kappa)^{-1}\cdot|T_n^\kappa|$.

Moreover, since
$$T_n^\kappa=nA_n^\kappa a_n^\kappa,$$
and
$$A_n^\kappa\sim\frac{n^\kappa}{\Gamma(\kappa+1)},\quad \text{for } \kappa\neq-1,-2,-3,\cdots,$$
we see that the summability $|C,\kappa|$ of $\sum a_n$ is the same as the convergence of
$$\sum n^{-(\kappa+1)}|na_n^\kappa A_n^\kappa|.$$
Similarly, according to our notations, the summability $|C,p|$ of

*) Kogbetliantz [3], [4].

$\sum \lambda_n a_n$ is the same as the convergence of the series

$$\sum n^{-(p+1)} |nA_n^p a_{n,\lambda}^p|.$$

We shall remember this for the analysis of the proof of our theorems. We use the notations:

$$\Delta\varphi_n = \Delta^1\varphi_n = \varphi_n - \varphi_{n+1},$$
$$\Delta^{p+1}\varphi_n = \Delta(\Delta^p\varphi_n) = \Delta^p\varphi_n - \Delta^p\varphi_{n+1}, \quad \text{for } p=1, 2, 3, \cdots.$$

We use the identity, for positive integral q,

$$\Delta^q\lambda_n = \lambda_n - \binom{q}{1}\lambda_{n+1} + \binom{q}{2}\lambda_{n+2} - \cdots + (-1)^q\lambda_{n+q}.$$

2. *Introduction.* By definition, the summability $|C, \kappa|$, $(\kappa > -1)$, of $\sum a_n$ is, of course, the same thing as the absolute convergence of the (C, κ)-transformed series $\sum a_n^\kappa$. The question naturally arises if there is a similar connection between the summability $|C, \kappa|$ of $\sum \lambda_n a_n$ and the absolute convergence of $\sum \lambda_n a_n^\kappa$. An answer has been given to this question by Hyslop in the case in which $\lambda_n = n^\rho$, $0<\rho<1$, in the form of the following theorem.

Theorem A.*) *If $0<\rho<1$, the necessary and sufficient condition that the series $\sum n^\rho a_n$ should be summable $|C, p|$, where p is a positive integer, is that the series $\sum n^\rho a_n^p$ is absolutely convergent.*

In the present paper, we treat the general case, and obtain an extended form of Theorem A, in the form of two theorems, corresponding respectively to the sufficiency and necessity parts of Hyslop's Theorem A.

3.1. We establish the following theorems.

THEOREM 1. *If λ_n is positive, monotonic increasing, tending to infinity with n, and $\Delta^q\lambda_n = O(n^{-q}\lambda_n)$, as $n\to\infty$, for $q=1, 2, 3, \cdots, p$, then the absolute convergence of $\sum \lambda_n a_n^p$ is the sufficient condition for the summability $|C, p|$ of $\sum \lambda_n a_n$, where p is a positive integer.*

THEOREM 2. *If λ_n is positive, monotonic increasing, tending to infinity with n, and satisfies the conditions:*

(i) $\qquad \Delta^q(\lambda_n^{-1}) = O(n^{-q}\lambda_n^{-1}),$ *as $n\to\infty$ for $q=1, 2, \cdots, p$;*

(ii) $\qquad \sum_{\kappa}^{\infty} n^{-1-\delta}\lambda_n = O(\kappa^{-\delta}\lambda_\kappa),$ *as $\kappa\to\infty$,*

then the absolute convergence of $\sum \lambda_n a_n^p$ is a necessary condition for the summability $|C, p|$ of $\sum \lambda_n a_n$ where p is a positive integer.

3.2. We shall require the following lemmas:

Lemma 1.**) *For $\kappa>0$, and $|x|<1$,*

$$\sum_{n=1}^{\infty} nA_n^\kappa a_n^\kappa x^n = (1-x)^{-\kappa} \sum_{n=0}^{\infty} na_n x^n,$$

assuming $\sum_{n=0}^{\infty} na_n x^n$ to be convergent for $|x|<1$.

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*) Hyslop [2].

**) Kogbetliantz [3].

Lemma 2.*) *If r is any real number and p a positive integer, then*

$$\sum_{\nu=\mu}^{n} \lambda_\nu A_{n-\nu}^{r} A_{\nu-\mu}^{-(p+1)} = \sum_{q=0}^{p} \binom{p}{q} A_{n-\mu-q}^{r+q-p} \Delta^q \lambda_\mu.$$

4.1. *Proof of Theorem* 1. By Lemma 1, for $|x|<1$,

$$\sum_{n=1}^{\infty} n A_n^p a_{n,\lambda}^p x^n = (1-x)^{-p} \sum_{n=0}^{\infty} n\lambda_n a_n x^n$$
$$= \left(\sum_{n=0}^{\infty} A_n^{p-1} x^n\right)\left(\sum_{n=0}^{\infty} n\lambda_n a_n x^n\right).$$

Therefore, on comparing the coefficients of x^n, which are justified by the absolute convergence of all the series concerned,

$$n A_n^p a_{n,\lambda}^p = \sum_{\nu=1}^{n} A_{n-\nu}^{p-1} \lambda_\nu (\nu a_\nu)$$
$$= \sum_{\nu=1}^{n} A_{n-\nu}^{p-1} \lambda_\nu \sum_{\mu=1}^{\nu} A_{\nu-\mu}^{-(p+1)} \mu A_\mu^p a_\mu^p.$$
$$= \left(\sum_{\mu=1}^{n-p} + \sum_{\mu=n-p+1}^{n}\right) \mu A_\mu^p a_\mu^p \sum_{\nu=\mu}^{n} \lambda_\nu A_{n-\nu}^{p-1} A_{\nu-\mu}^{-(p+1)}$$
$$= E_1 + E_2, \quad \text{say.}$$

Now,**) $$\sum n^{-(p+1)} |E_2| \le K \sum_n n^{-(p+1)} n A_n^p |a_n^p| \lambda_n$$
$$\le K \sum_n |a_n^p| \lambda_n$$
$$\le K < \infty.$$

We proceed to show that

$$\sum n^{-(p+1)} |E_1| < \infty.$$

By Lemma 2,

$$E_1 = \sum_{\mu=1}^{n-p} \mu A_\mu^p a_\mu^p \sum_{\nu=\mu}^{n} \lambda_\nu A_{n-\nu}^{p-1} A_{\nu-\mu}^{-(p+1)}$$
$$= \sum_{q=0}^{p} \binom{p}{q} F_q,$$

where $$F_q = \sum_{\mu=1}^{n-p} \mu A_\mu^p a_\mu^p A_{n-\mu-q}^{q-1} \Delta^q \lambda_\mu \quad (q=1,2,\cdots,p).$$

When $q=0$, $F_q=0$, since $A_{n-\mu}^{-1}=0$.

Now $$\sum_n n^{-(p+1)} |F_q| \le K \sum_n n^{-(p+1)} \left|\sum_{\mu=1}^{n-p} \mu A_\mu^p a_\mu^p A_{n-\mu-q}^{q-1} \Delta^q \lambda_\mu\right|.$$

Now, by hypothesis, we have

$$\sum_n n^{-(p+1)} |F_q| \le K \sum_n n^{-(p+1)} \sum_{\mu=1}^{n-p} \mu^{p+1} |a_\mu^p| A_{n-\mu-q}^{q-1} \mu^{-q} \lambda_\mu$$
$$\le K \sum_{n=p+1}^{\infty} n^{-(p+1)} \sum_{\mu=1}^{n-p} \mu^{p-q+1} \lambda_\mu |a_\mu^p| (n-\mu-q+1)^{q-1}$$
$$= K \sum_{\mu=1}^{\infty} \mu^{p-q+1} \lambda_\mu |a_\mu^p| \sum_{n=\mu+p}^{\infty} n^{-(p+1)} (n-\mu-q+1)^{q-1}$$
$$\le K \sum_{\mu=1}^{\infty} \mu^{p-q+1} \lambda_\mu |a_\mu^p| \sum_{n=\mu+p}^{\infty} n^{-p+q-2}$$

*) Andersen [1]; Hyslop [2].

**) Here, as elsewhere, K denotes an absolute constant, not necessarily the same at each occurrence.

$$\leq K\sum_{\mu=1}^{\infty}\mu^{p-q+1}\lambda_\mu|a_\mu^p|\mu^{-p+q-1}$$
$$=K\sum_{\mu=1}^{\infty}\lambda_\mu|a_\mu^p|$$
$$\leq K<\infty.$$

Thus $\sum n^{-(p+1)}|E_1|<\infty$ $(p=1,2,\cdots)$, and $\sum n^{-(p+1)}|nA_n^p a_{n,\lambda}^p|<\infty$, that is $\sum\lambda_n a_n$ is summable $|C,p|$, and this completes the proof of Theorem 1.

4.2. *Proof of Theorem* 2. We assume the summability $|C,p|$ of $\sum\lambda_n a_n$ that is, that $\sum|a_{n,\lambda}^p|<\infty$, and we have to show that $\sum\lambda_n|a_n^p|<\infty$, or what is the same thing,

$$\sum\lambda_n n^{-p-1}|nA_n^p a_n^p|<\infty.$$

$$nA_n^p a_n^p=\left(\sum_{\mu=1}^{n-p}+\sum_{\mu=n-p+1}^{n}\right)\mu A_\mu^p a_{\mu,\lambda}^p\sum_{\nu=\mu}^{n}(\lambda_\nu)^{-1}A_{n-\nu}^{p-1}A_{\nu-\mu}^{-(p+1)}$$
$$=E_1'+E_2',\quad\text{say.}$$

Now,
$$\sum_n\lambda_n n^{-p-1}|E_2'|\leq K\sum\lambda_n n^{-p-1}|n\cdot n^p\cdot a_{n,\lambda}^p(\lambda_n)^{-1}|$$
$$\leq K\sum_n\lambda_n(\lambda_n)^{-1}|a_{n,\lambda}^p|$$
$$=\sum_n|a_{n,\lambda}^p|\leq K<\infty.$$

We proceed to show that

$$\sum\lambda_n n^{-p-1}|E_1'|<\infty$$

$$E_1'=\sum_{\mu=1}^{n-p}\mu A_\mu^p a_{\mu,\lambda}^p\sum_{\nu=\mu}^{n}(\lambda_\nu)^{-1}A_{n-\nu}^{p-1}A_{\nu-\mu}^{-(p+1)}$$
$$=\sum_{\mu=1}^{n-p}\mu A_\mu^p a_{\mu,\lambda}^p\sum_{q=0}^{p}\binom{p}{q}A_{n-\mu-q}^{q-1}\Delta^q(\lambda_\mu^{-1})$$
$$=\sum_{q=0}^{p}\binom{p}{q}F_q',$$

where

$$F_q'=\sum_{\mu=1}^{n-p}\mu A_\mu^p a_{\mu,\lambda}^p A_{n-\mu-q}^{q-1}\Delta^q(\lambda_\mu)^{-1}.$$

Now,

$$\sum_{n=p+1}^{\infty}\lambda_n n^{-p-1}|F_q'|=\sum_{n=p+1}^{\infty}\lambda_n n^{-p-1}\left|\sum_{\mu=1}^{n-p}\mu A_\mu^p a_{\mu,\lambda}^p A_{n-\mu-q}^{q-1}\Delta^q(\lambda_\mu)^{-1}\right|$$
$$\leq K\sum_{n=p+1}^{\infty}\lambda_n n^{-p-1}\sum_{\mu=1}^{n-p}\mu\mu^p|a_{\mu,\lambda}^p|A_{n-\mu-q}^{q-1}\mu^{-q}(\lambda_\mu)^{-1},$$

(by hypothesis (i) of Theorem 2),

$$\leq K\sum_{\mu=1}^{\infty}\mu^{1+p-q}(\lambda_\mu)^{-1}|a_{\mu,\lambda}^p|\sum_{n=\mu+p}^{\infty}\lambda_n n^{-p-1}n^{q-1}$$
$$\leq K\sum_{\mu=1}^{\infty}\mu^{1+p-q}(\lambda_\mu)^{-1}|a_{\mu,\lambda}^p|\sum_{n=\mu}^{\infty}n^{-p+q-2}\lambda_n$$
$$\leq K\sum_{\mu=1}^{\infty}\mu^{1+p-q}(\lambda_\mu)^{-1}|a_{\mu,\lambda}^p|\mu^{-p+q-1}\lambda_\mu,$$

(by hypothesis (ii) of Theorem 2),

$$=K\sum_{\mu=1}^{\infty}|a_{\mu,\lambda}^p|\leq K<\infty,$$

by hypothesis.

This completes the proof of the theorem.

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