49. *On a Criterion of Quasi-boundedness of Positive Harmonic Functions*

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1. For a positive[1] harmonic function u on a Riemann surface R, we denote by $\mathfrak{B}u$ the positive harmonic function on R defined by

$$(\mathfrak{B}u)(p)=\sup(v(p);\ u\geqq v,\ v\in HB(R))$$

for p in R. After Parreau we say that u is *quasi-bounded* if $\mathfrak{B}u=u$. In this note we shall give a condition for a positive harmonic function to be quasi-bounded by using the rate of diminishing of harmonic measures of level curves of the harmonic function. For the aim, we set

$$\mathfrak{L}(u;a)=(p\in R;\ u(p)=a)$$

for any positive number a. This is the *a-level curve* of u. For any closed subset F of R, we denote

$$\omega(F;p)=\inf s(p),$$

where s runs over all positive superharmonic functions on R such that $s\geqq 1$ on F. This is the *harmonic measure* of F relative to R calculated at p. Now fix a point p in R. It is clear that $\omega(\mathfrak{L}(u;a);p)=O(1/a)$ for $a\to\infty$. If u is bounded, then $\omega(\mathfrak{L}(u;a);p)=0$ for $a>\sup u$. This suggests us that $\omega(\mathfrak{L}(u;a);p)=o(1/a)$ might be a condition for u to be quasi-bounded. This is really the case and we shall prove

Theorem. *For a positive harmonic function u on a Riemann surface R, the following three conditions are mutually equivalent:*

(1) *u is quasi-bounded on R;*

(2) $\lim_{a\to\infty} a\omega(\mathfrak{L}(u;a);p)=0$ *for some (and hence for any) point p in R;*

(3) $\liminf_{a\to\infty} a\omega(\mathfrak{L}(u;a);p)=0$ *for some (and hence for any) point p in R.*

2. It is clear that the condition (2) implies the condition (3). Hence we have only to show the implications (1)$\to$(2) and (3)$\to$(1). In each case, we may assume that u is unbounded on R and $R\notin O_{HP}$.

Proof of the implication (1)$\to$(2). Fix a point p in R and let R_a be the connected component of the open set $(q\in R;\ u(q)<a)(a>u(p))$ containing the point p. Clearly $\bigcup_{a>u(p)} R_a=R$. Let R^* be the Wiener compactification[2] of R, $\Delta=R^*-R$ and μ be the harmonic measure[2] on

1) By positive, we mean non-negative.

2) C. Constantinescu-A. Cornea: Ideale Ränder Riemannscher Flächen. Springer (1963).

Δ with the reference point p. We denote by $\bar{R}_a$ the closure of R_a in R^* and set $\Delta_a=\Delta\cap\bar{R}_a$. Then $\Delta_a\cup(\partial R_a)=\bar{R}_a-R_a$ and $\bar{R}_a$ is clearly a resolutive compactification[2] of R_a. So we can speak of the harmonic measure[2] $\bar{\mu}_a$ on $\Delta_a\cup(\partial R_a)$ with the reference point p. We set

$$\mu_a=\begin{cases}\bar{\mu}_a & \text{on } \Delta_a;\\ 0 & \text{on } \Delta-\Delta_a.\end{cases}$$

Then μ_a is the measure on Δ with $0\leqq\mu_a\leqq\mu_{a'}\leqq\mu$ for $a<a'$. Let $\Delta_\infty=\bigcup_{a>u(p)}\Delta_a$. Then $u(q)=\infty$ on $\Delta-\Delta_\infty$ and so

$$u(p)=\int_\Delta u(q)d\mu(q)$$

shows that $\mu(\Delta-\Delta_\infty)=0$. From this it easily follows that

$$(*)\qquad \lim_{a\to\infty}\int_\Delta v(q)d\mu_a(q)=\int_\Delta v(q)d\mu(q)$$

for any v in $HB(R)$. Let u_n be the least harmonic majorant of u and n. Then $u_n(q)=\min(u(q),n)$ on Δ μ-almost everwhere. Clearly

$$\limsup_{a\to\infty}\int_\Delta u(q)d\mu_a(q)\leqq\int_\Delta u(q)d\mu(q).$$

On the other hand,

$$\liminf_{a\to\infty}\int_\Delta u(q)d\mu_a(q)\geqq\lim_{a\to\infty}\int_\Delta u_n(q)d\mu_a(q).$$

By using $(*)$ and by letting $n\nearrow\infty$, we get

$$\liminf_{a\to\infty}\int_\Delta u(q)d\mu_a(q)\geqq\int_\Delta u(q)d\mu(q).$$

Hence we finally conclude that

$$(**)\qquad \lim_{a\to\infty}\int_\Delta u(q)d(\mu-\mu_a)(q)=0.$$

Now it is easy to see that

$$\omega(\mathfrak{L}(u;a);p)=(1/a)u(p)-(1/a)\int_\Delta u(q)d\mu_a(q)$$
$$=(1/a)\int_\Delta u(q)d(\mu-\mu_a)(q).$$

This with $(**)$ gives that $\lim_{a\to\infty} a\omega(\mathfrak{L}(u;a);p)=0$.

Proof of the implication (3)→(1). Choose a sequence (a_n) of positive numbers such that $u(p)<a_1<a_2<\cdots<a_n<\cdots$, $\lim_{n\to\infty}a_n=\infty$ and $a_n\omega(\mathfrak{L}(u;a_n);p)<1/n^3$. We set $b_n=na_n$ and

$$w_n(q)=\begin{cases}\omega(\mathfrak{L}(u;a_n);q) & q\in R_{a_n};\\ 1 & q\in R-R_{a_n}\end{cases}$$

and

$$w(q)=\sum_{n=1}^\infty b_nw_n(q)$$

on R. Then clearly $w(q)$ is a positive superharmonic function on R and so continuous on the Wiener compactification of R.[2] Let

$$h(q)=u(q)-(\mathfrak{B}u)(q),$$

which is a positive harmonic function on R. We have to show that $h\equiv0$ on R. On the Wiener harmonic boundary[2] Γ of R, h vanishes.[2]

Hence it follows that

$$0 \leqq h(q) \leqq u(q) \leqq a_n = b_n/n \leqq w(q)/n$$

on $(\bar{R}_{a_n} \cap \Gamma) \cup (\partial R_{a_n})$. By a maximum principle,[2] we get

$$0 \leqq h(q) \leqq w(q)/n$$

on R_{a_n}. Hence by making $n \nearrow \infty$, we must have $h \equiv 0$ on R.

3. For an example, consider the circular slits disc

$$R = (z;\, 0 < |z| < 1) - \cup_{n=1}^{\infty} \mathfrak{S}_n.$$

Here $\mathfrak{S}_n$ is a circular slit $(r_n e^{i\theta};\, -\alpha_n \leqq \theta \leqq \alpha_n)$, where

$$1 > r_1 > r_2 > \cdots > r_n > \cdots,\ \lim_{n \to \infty} r_n = 0$$

and $0 \leqq \alpha_n < \pi$.

If we take α_n so large as to make the harmonic measure of the complementary circular slit $(r_n e^{i\theta};\, \alpha_n \leqq \theta \leqq 2\pi - \alpha_n)$ with respect to $(z;\, |z| < 1)$ less larger than $\varepsilon_n / \log(1/r_n)$ with $\varepsilon_n \searrow 0$, then $\log(1/|z|)$ is quasi-bounded on R. This is easily seen by checking the condition (3).

Contrary, we can choose α_n so small as to make the harmonic measure of $(re^{i\theta};\, 0 \leqq \theta \leqq 2\pi)$ with respect to R less smaller than $1/2 \log(1/r)$ for any r in $(0, 1)$. The extreme case of such a type is obtained by taking $\alpha_n = 0$ $(n = 1, 2, \cdots)$. In such a case, $\log(1/|z|)$ is not quasi-bounded on R. This is readily seen from the condition (2).