4. *Connection of Topological Fibre Bundles*

By Akira ASADA

(Comm. by Kinjirô KUNUGI, M.J.A., Jan. 12, 1966)

In Asada [2], we give a general theory of connections of topological vector bundles. There a connection form $\{\theta_U\}$ of the given bundle ξ has following property: The value of $1+\theta_U$ belongs in G, the structure group of ξ. Therefore starting from $\{s_U\}=\{1+\theta_U\}$, we can construct a theory of connections of arbitrary topological fibre bundles without using the ring A of Asada [2]. To state this theory is the purpose of this note. But we don't know whether there exists or not a connection form for an arbitrary fibre bundle ξ.

1. *Connection of fibre bundles.* We denote by X a topological space, ξ a topological fibre bundle over X with structure group G. The transition functions of ξ are denoted by g_{UV}.

As in Asada [2], $n^\circ 1$, we denote the group of continuous maps from $V(\Delta_s(U))$ to G with equivalence relation $f_1\sim f_2$ if and only if $f_1| W=f_2| W$ for some neighborhood $W(\Delta_s(U))$ of $\Delta_s(U)$ in $U\times\cdots\times U$ by $\tilde{C}^s(U, G)$ and set

$C^s(U, G)=\{f\mid f\in\tilde{C}^s(U, G), f(\cdots, x_i, x_i, \cdots)=1$ for all i, $0\leq i\leq s-1\}$.

Then we define the sheaves $\tilde{\mathcal{G}}^r=\tilde{\mathcal{G}}^r(\xi)$ and $\mathcal{G}^r=\mathcal{G}^r(\xi)$ by

$\tilde{\mathcal{G}}^r$: the sheaf of germs of those maps $\{f_U\}$, $f_U\in\tilde{C}^r(U, G)$,
$g_{UV}(x_0)^{-1}f_U(x_0, \cdots, x_r)g_{UV}(x_r)=f_V(x_0, \cdots, x_r)$.

$\mathcal{G}^r$: the subsheaf of $\tilde{\mathcal{G}}^r$ consisted those elements $\{f_U\}$ that $f_U\in C^r(U, G)$ for all U.

Definition. If $\{s_U\}\in H^0(X, \mathcal{G}^1)$, then we call $\{s_U\}$ is a connection form of ξ.

Note. As usual, if $\{s_U\}$ is a connection form of $\{g_{UV}\}$, $\{U'\}$ is a refinement of $\{U\}$ and $g_{U'V'}=g_{UV}\mid U'\cap V'$, then $\{s_{U'}\}$, $s_{U'}=s_U\mid U'$, becomes a connection form of $\{g_{U'V'}\}$. We identify $\{s_U\}$ and this $\{s_{U'}\}$. On the other hand, if $\{s_U\}$ is a connection form of $\{g_{UV}\}$ then $\{h_U(x_0)s_U(x_0, x_1)h_U(x_1)^{-1}\}$ is a connection form of $\{h_U g_{UV} h_V^{-1}\}$. We identify $\{s_U\}$ and this $\{h_U s_U h_U^{-1}\}$. For the simplicity, we identify $\{s_U\}$ and the equivalence class of $\{s_U\}$.

Lemma 1. *$H^0(X, \mathcal{G}^1)$ is non-empty if and only if $H^0(X, \tilde{\mathcal{G}}^1)$ is non-empty.*

Lemma 2. *$\{1\}$ belongs in $H^0(X, \mathcal{G}^1)$ if and only if $\{1\}$ belongs in $H^0(X, \tilde{\mathcal{G}}^1)$.*

Theorem 1. *ξ is equivalent to a bundle with tatally disconnect structure group if and only if $\{1\}$ becomes a connection form of ξ.*

Proof. If $\{h_U g_{UV} h_V^{-1}\}$ is locally constant, then $\{s_U(x_0, x_1)\}=$

$\{h_U(x_0)h_U(x_1)^{-1}\}$ belongs in $H^0(X, \mathcal{Q}^1)$. On the other hand, if $\{1\}$ belongs in $H^0(X, \mathcal{Q}^1)$, then $\{g_{UV}\}$ is locally constant. Hence we get the theorem.

Corollary. *The connected component of the structure group of ξ is reduced to H, a subgroup of G, if there exists a connection form $\{s_U\}$ of ξ such that the class of the value of $\{s_U\}$ in $H\backslash G/H$ is equal to* 1.

2. *Elements of $C^1(X_G, G)$ derived from connection forms.* We denote by X_G the associated principal bundle of ξ. The projection from X_G to X is denoted by $\pi=\pi_G$. The homeomorphism from $\pi^{-1}(U)$ to $U\times G$ is denoted by $\varphi_U\cdot\varphi^{-1}(x, a)$ is denoted by $\varphi_U^{-1}(x)(a)$. Then setting

$$\alpha c=\varphi_U^{-1}(x)(ac),\ \alpha=\varphi_U^{-1}(x)(a)\in X_G,\ c\in G,$$

G operates on X_G.

Theorem 2. *We set*

$$(1)\qquad C^1(X_G, G)_G=\{s \mid s\in C^1(X_G, G),\ s(\alpha a, \beta b)=a^{-1}s(\alpha, \beta)b,\ \alpha, \beta\in X_G,\ a, b\in G\}.$$

Then there is a bijection $t=t_G$ between $H^0(X, \mathcal{Q}^1)$ and $C^1(X_G, G)_G$.

Proof. If $\{s_U\}$ belongs in $H^0(X, \mathcal{Q}^1)$, then we set

$$t(\{s_U\})(\varphi_U(x)^{-1}(a), \varphi_U(y)^{-1}(b))=a^{-1}s_U(x, y)b.$$

By the definition of $\mathcal{Q}^1$, this definition of t does not depend on the choice of U and $t(\{s_U\})$ belongs in $C^1(X_G, G)_G$. On the other hand, if s belongs in $C^1(X_G, G)_G$, then setting

$$t^{-1}(s)_U(x, y)=as(\varphi_U^{-1}(x)(a), \varphi_U^{-1}(y)(b))b^{-1},$$

$t^{-1}(s)_U(x, y)$ does not depend on the choice of a, b and $\{t^{-1}(s)_U\}$ belongs in $H^0(X, \mathcal{Q}^1)$. Moreover, we have $tt^{-1}(s)=s$, $t^{-1}t(\{s_U\})=\{s_U\}$. Hence we obtain the theorem.

Corollary. *Setting*

$$T^1(X_G, G)=\{r \mid r\in C^1(X_G, G),\ r(\alpha a, \beta b)=a^{-1}r(\alpha, \beta)a\},$$

we get

$$(2)\qquad C^1(X_G, G)_G=T^1(X_G, G)s,\ s\in C^1(X_G, G)_G,$$

if $C^1(X_G, G)_G\neq\varnothing$.

Note. Similarly, we can prove that there is a bijection t_H between $H^0(X_{G/H}, \mathcal{Q}^1)$ and $C^1(X_G, G)_H$ for arbitrary subgroup H of G. Here $X_{G/H}$ is the associated G/H-bundle of ξ, $H^0(X_{G/H}, \mathcal{Q}^1)$ is the set of connections of $\pi_{G/H}{}^*(\xi)$ and $C^1(X_G, G)_H$ is given by

$$C^1(X_G, G)_H=\{s \mid s\in C^1(X_G, G),\ s(\alpha a, \beta b)=a^{-1}s(\alpha, \beta)b,\ \alpha, \beta\in X_G,\ a, b\in H\}.$$

By theorem 1 and above note, we obtain (cf. [3]),

Theorem 4′. *X_G is induced from a covering space of $X_{G/H}$ if and only if $\{1\}$ belongs in $t_e^{-1}(C^1(X_G, G)_H)$. Here t_e means the bijection from $H^0(X_G, \mathcal{Q}^1)$ to $C^1(X_G, G)$.*

3. *Curvature form of a connection form.* **Lemma 3.** X_G has a cross-section from X if and only if there exists a continuous function $s\colon X_G \to G$ such that

(3) $$s(\alpha a)=s(\alpha)a,\ \alpha\in X_G,\ a\in G.$$

Lemma 4. *If $s_1, s_2\colon X_G\to G$ satisfy* (2), *then setting*

(4) $$s(\alpha, \beta)=s_1(\alpha)^{-1}s_2(\beta),$$

$s(\alpha, \beta)$ becomes a connection form of X.

Since G is a non-abelian group in general, although $C^r(X_G, G)$ and $\delta_r\colon C^r(X_G, G)\to C^{r+1}(X_G, G)$ are defined for all r, $\delta_{r+1}\delta_r\neq 1$ if $r\geq 1$ in general. But $\delta_1\delta_0$ is equal to 1 for all G. Strictly, $\delta_1 f$ is given by

$$(\delta_1 f)(x_0, x_1, x_2)=f(x_1, x_2)f(x_0, x_2)^{-1}f(x_0, x_1).$$

We denote ker. δ_r by $Z^r(X_G, G)$.

Definition. If s belongs in $C^1(X_G, G)_G$, then we call $\delta_1 s$ the curvature form of s.

By definition, if s is a connection form, then we get

(5) $$(\delta_1 s)(\alpha a, \beta b, \gamma c)=b^{-1}(\delta_1 s)(\alpha, \beta, \gamma)b.$$

Note. We define the sheaf $\mathcal{G}^{(2)}=\mathcal{G}^{(2)}(\xi)$ as follows.

$\mathcal{G}^{(2)}$: the sheaf of germs of those maps $\{f_U\}$, $f_U\in C^2(U, G)$ and

$$g_{UV}(x_1)^{-1}f_U(x_0, x_1, x_2)g_{UV}(x_1)=f_V(x_0, x_1, x_2).$$

Then we can define the map δ from $H^0(X, \mathcal{G}^1)$ to $H^0(X, \mathcal{G}^{(2)})$ and we call $\delta(\{s_U\})$ is the curvature form of $\{s_U\}$. Moreover, the following diagram is commutative

$$\begin{array}{ccc} C^1(X_G, G)_G & \xrightarrow{\delta_1} & T^2(X_G, G) \\ {\scriptstyle t}\uparrow & & \uparrow{\scriptstyle t'} \\ H^0(X, \mathcal{G}^1) & \xrightarrow{\delta} & H^0(X, \mathcal{G}^{(2)}), \end{array}$$

where $T^2(X_G, G)$ and t' are given by

$$T^2(X_G, G)=\{r \mid r\in C^2(X_G, G),\ r(\alpha a, \beta b, \gamma c)=b^{-1}r(\alpha, \beta, \gamma)b\},$$

$$t'(\{f_U\})(\varphi_U(x)(a), \varphi_U(y)(b), \varphi_U(z)(c))=b^{-1}f_U(x, y, z)b.$$

Lemma 5. *If η is an H-bundle and H is a subgroup of G, then to denote i the inclusion from H to G, we have the following commutative diagram.*

$$\begin{array}{ccc} H^0(X, \mathcal{G}^1(i^*(\eta))) & \xrightarrow{\delta} & H^0(X, \mathcal{G}^{(2)}(i^*(\eta))) \\ {\scriptstyle i^*}\uparrow & & \uparrow{\scriptstyle i^*} \\ H^0(X, \mathcal{G}^1(\eta)) & \xrightarrow{\delta} & H^0(X, \mathcal{G}^{(2)}(\eta)). \end{array}$$

Theorem 3. (*cf.* [2], *theorem* 2, [7], [8]). *If the value of a curvature form of ξ belongs in H, a subgroup of G, then the connected component of the structure group of ξ is reduced to H.*

Corollary. *If ξ has a connection form which is a cocycle, then X_G is induced from a representation of $\pi_1(X)$ in G.*

Similarly, if we use the notion of curvature forms, theorem 4′ is refined as follows.

Theorem 4. *X_G is induced from a covering space of $X_{G/H}$, where H is a subgroup of G, if and only if 1 belongs in $\delta_1(C^1(X_G, G)_H)$, or equivalently*

$$C^1(X_G, G)_H \cap Z^1(X_G, G) \neq \emptyset. \tag{6}$$

4. *Associated vector bundle of a topological fibre bundle.* To show the relations between above theory of connections and the theory given in [2], first we construct an associated vector bundle of ξ.

We denote by F the fibre of ξ and assume that F has sufficiently many continuous functions. We denote by $C(F)$ the topological vector space over $\boldsymbol{R}$ consisted by all real valued continuous functions on F with compact open topology. The ring of all linear operators of $C(F)$ is denoted by $R(C(F))$.

For $T \in R(C(F)), f_1, \cdots, f_n \in C(F)$, a compact set K of F, we set

$$U(T, f_1, \cdots, f_n, K, \varepsilon)=\{S \mid |(Tf_i)(x)-(Sf_i)(x)|<\varepsilon,\ x \in K,\ 1 \leq i \leq n\}.$$

Then taking $\{U(T, f_1, \cdots, f_n, K, \varepsilon)\}$ to be open basis of $R(C(F))$, $R(C(F))$ becomes a topological ring. ([6], § 33).

Lemma 6. *For $a \in G, f \in C(F)$, we set*

$$(\iota(a)f)(x)=f(a^{-1}(x)). \tag{7}$$

Then $\iota(a)$ belongs in $R(C(F))$ and the map $\iota: G \rightarrow R(C(F))$ is a continuous monomorphism.

Note. ι is not a homeomorphism in general. But we obtain

Lemma 7. *We assume that F is a C^r-class manifold and G is a group of C^r-diffeomorphisms of F with $C^{r'}$-topology. $(r' \leq r)$. (cf. [1]). Here C^0-manifold, C^0-diffeomorphism, and C^0-topology mean topological manifold, homeomorphism and compact open topology. The topological vector space consisted by all $C^{r'}$-class functions on F with $C^{r'}$-topology is denoted by $C^{r'}(F)$ and the ring of all linear operators of $C^{r'}(F)$ is denoted by $R(C^{r'}(F))$. Then if we take*

$$U(T, f_1, \cdots f_n, K, \varepsilon)$$
$$=\{S \mid |D^p(Tf_i)(x)-D^p(Sf_i)(x)|<\varepsilon,\ x \in K,\ |p| \leq r',\ 1 \leq i \leq n\},$$
$$p=(j_1, \cdots, j_n),\ |p|=j_1+\cdots+j_n,\ D^p=\frac{\partial^{|p|}}{\partial x_1^{j}1 \cdots \partial x_n^{j}n},\ D^0 f=f,$$

to be open basis of $R(C^{r'}(F))$, the map $\iota: G \rightarrow R(C^{r'}(F))$ defined by (7) is a homeomorphism.

In the rest, we denote the associated $C(F)$-bundle ($C^{r'}(F)$-bundle) of ξ by $v(\xi)$ $(v^{r'}(\xi))$, then by lemma 7, if ξ is an $H_0(n)$-bundle $(H_0^{r,r'}(n)$-bundle (cf. [1])) then the correspondence $\xi \rightarrow v(\xi)$ $(\xi \rightarrow v^{r'}(\xi))$ is a bijection.

5. *Relations between the connections of this note and that of* [2]. Since $v(\xi)$ is a vector bundle, we can define its s-cross-sections. ([2], $n°2$). We can also define the addition of the elements of G and we get

Theorem 5. *If $\{s_U\}$ is a connection form of ξ, then setting $\{\theta_U\}=\{s_U-1\}$, we have*

$$(8) \qquad (d+\theta_U)f_U=\iota(g_{UV})(d+\theta_V)f_V,$$

for all $f_U \in C^s(X, v(\xi))$, $s \geq 0$. *Conversly, if the collection* $\{\theta_U\}$, $\theta_U \in C^1(U, R(C(F)))$ *satisfies* (8) *and the value of* $1+\theta_U$ *belongs in* $\iota(G)=G$, *then setting* $s_U=1+\theta_U$, $\{s_U\}$ *belongs in* $H^0(X, \mathcal{G}^1)$ *if* ι *is a homeomorphism.*

Note. By the proof of theorem 1 of [2], we know that if X is a paracompact normal topological space, then there always exists a collection $\{\theta_U\}$, $\theta_U \in C^1(U, R(C(F)))$ which satisfies (8).

Since we obtain

$$(9) \qquad d\theta_U+\theta_U\theta_U=s_U(x_0, x_1)s_U(x_1, x_2)-s_U(x_0, x_2),$$

if $\{\theta_U\}=\{s_U-1\}$, the definition of the curvature form of a connection form must different from the definition of this note if we use the definition of curvature forms of [2]. But theorem 3 of this note and theorem 2 of [2] show that there must be relations between curvature forms defined by the right hand side of (9) and defined as $\delta_1(t(\{s_U\}))$. For example, we obtain

$$(10) \qquad \delta_1(t(\{s_U\}))(x_0, x_1, x_2)=1 \text{ if and only if } s_U(x_0, x_1)s_U(x_1, x_2)-s_U(x_0, x_2)=0.$$

Definition. $\{\Theta_U\}=\{d\theta_U+\theta_U\theta_U\}$ is called a G-valued curvature form if the value of $1+\Theta_U$ belongs in G.

By lemma 5, theorem 3 and theorem 2 of [2], the definition that a curvature form to be G-valued does not depend on the definitions of curvature form.

6. *Connection of microbundles and topological manifolds.* By Kister's theorem ([4]), a topological microbundle $\mathfrak{X}$([5]) over a locally finite complex is induced from a unique $H_0(n)$-bundle over X. Here $H_0(n)$ is the group of all homeomorphisms of R^n which fix the origin with compact open topology. Therefore we consider $H_0(n)$-bundles over X instead of microbundles over X. Then according to the definitions of this note, we can consider the connections of microbundles.

Definition. If X is a topological manifold, then a connection of the tangent microbundle of X ([5]) is called a connection of X.

Although we don't know the existence of connections for microbundles, a $GL(n, R)$-bundle always has a connection form if X is a paracompact normal topological space. ([2], $n°2$). Therefore we obtain by lemma 5 and theorem 3, (or theorem 2 of [2]),

Theorem 6. *An* $H_0(n)$*-bundle* $\mathfrak{X}$ *over a simply connected paracompact normal topological space is induced from a vector bundle if and only if* $\mathfrak{X}$ *has a connection form with matrix valued curvature form.*

Corollary. *A simply connected PL-manifold X can be given a smoothness structure if and only if X has a connection form* *with matrix valued curvature form.*

This follows from theorem 6 and [5], theorem (5, 12).

Note. By lemma 7, the correspondence $\mathfrak{X}\to v(\mathfrak{X})$ is a bijection. But if we use $C'(R^n)$, the Banach space consisted by the all bounded continuous functions on R^n with norm $\|f\|=\max_{x\in R^n}|f(x)|$, instead of $C(R^n)$ and use the strong topology of $R(C'(R^n))$, the ring of all linear operators of $C'(R^n)$, then ι: $H_0(n)\to R(C'(R^n))$ is not continuous. In fact, we obtain

$$\|\iota(a)-\iota(b)\|\geq 1,$$

for all $a, b\in H_0(n)$.

References

[1] Asada, A.: Contraction of the group of diffeomorphisms of R^n. Proc. Japan Acad., **41**, 273-276 (1965).

[2] ——: Connection of topological vector bundles. Proc. Japan Acad., **41**, 919-922 (1965).

[3] Hirzebruch, F.: Neue Topologische Methoden in der Algebraische Geometry. Berlin (1956).

[4] Kister, J. M.: Microbundles are fibre bundles. Ann. of Math., **80**, 190-199 (1964).

[5] Milnor, J.: Microbundles, Part I. Topology, **3**, 53-80 (1964).

[6] Neumark, M. A.: Normierte Algebren. Berlin (1959).

[7] Nomizu, K.: Un théorèm sur les group d'holonomie. Nagoya Math. J., **10**, 101-103 (1956).

[8] ——: Lie Groups and Differential Geometry. Tokyo (1956).