69. *Some Properties of* $\sum_1^1$- *and* $\prod_1^1$-*sets in* N^N

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Introduction. Ljapunow says that there exists a strong connection between the methods of the theory of recursive functions and of the descriptive set theory, and hence the time giving a new expression of the latter theory will present itself ("Einleitung" of [8]). Many authors realize this. Addison calls it the effective descriptive set theory. In this paper, we shall state some results concerning this field. Our main aim is to give the positions in the Kleene analytic hierarchy for sets and ordinals whose existence had been classically proved. When we say a set, it is a subset of N^N so long as we do not mention otherwise, where N^N denotes the set of all 1-place number-theoretic functions and is identified with the Baire zero-space in the usual manner. In our proofs, we shall often use of the following

Effective choice principle. *From every non-empty* $\prod_1^1$-*set (with respect to the Kleene hierarchy) we can elect a point* α_0 *such that the unit subset* $\{\alpha_0\}$ *also belongs to the same class; and hence* α_0 *itself can be expressed in the* Δ_2^1-*form (namely, in both the forms* $\sum_2^1$ *and* $\prod_2^1$) *as a number-theoretic function* [3; 10; 13].

The method using this principle is suggested by Sampei [11].

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§1. It is well-known that every non-denumerable analytic set (e.g. in the Baire zero-space) contains a non-empty perfect subset. (Cf. e.g., [8].) For any non-denumerable $\sum_1^1$-set E, the following question does arise: In what class a non-empty perfect subset of E can be found? For this, first we can prove the

Theorem 1. *For any recursive* R, $|\hat{\beta}(\exists\alpha)(x)R(\bar{\beta}(x),\bar{\alpha}(x))|>\aleph_0$ *can be expressed in the* $\sum_1^1$-*form.*

Using this theorem we obtain the

Theorem 2. *For any non-denumerable* $\sum_1^1$-*set* E *there exists a non-empty perfect subset* P *of* E *which is O-recursively closed.*[1)]

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1) This means that the complement of P is expressible in the form: $CP=\cup\{w_n \mid n\in Q\}$ for some Q recursive in O, where O is the Π_1^1-set of natural numbers defined in Kleene [2] and $\{w_n\}_{n=0,1,2,\cdots}$ is a recursive enumeration of all sequence numbers. Here we identify sequence numbers with Baire's intervals in the usual manner.

A fortiori, P belongs to the class Δ_2^1.

But, in general such a set P can not be in the class $\prod_1^1$, because we have the

Theorem 3. *There exists a non-denumerable* $\sum_1^1$*-set which contains no non-void* $\prod_1^1$*-subsets.*

Such a set is similar to a *simple* set of natural numbers in the sense of L. E. Post. Its construction is easily carried out by means of the effective choice principle. By the same way, we can obtain the

Theorem 4. *For each* $n\geq 2$, *there exists a non-denumerable* $\prod_n^1$*-set which contains no non-void* $\sum_n^1$*-subsets, where for* $n\geq 3$ *Gödel's axiom of constructibility is assumed.*

In the proof for $n\geq 3$, a result of Addison [1] is used.

§ 2. Let E be a $\sum_1^1$-set. We shall denote the condensation points of E contained in E by E_0. Let $E_1=E-E_0$. Then by using Theorem 1 we can easily prove the following

Theorem 5. *The* E_0 *is also a* $\sum_1^1$*-set,*[2] *and hence* E_1 *is a* $(\sum_1^1)_\rho$*-set; that is, it is the difference of two* $\sum_1^1$*-sets.*

The evaluation of E_1 can not be improved, because we have the

Theorem 6. *There is a* $\sum_1^1$*-set* E *for which the* E_1 *can not be contained in the class* $C(\sum_1^1)_\rho$.

Kondô [4] shows that if an analytic set is denumerable then it is effectively denumerable. As an effective version for this we have the following form:

Theorem 7. *Let* E *be a non-empty denumerable* $\sum_1^1$*-set. Then the members of* E *can be enumerated by a* Δ_2^1 *number-theoretic function.*

§ 3. Lusin [9] showed that one can choose a non-void perfect subset in a CA-set (i.e., complement of an analytic set) having a non-denumerable constituent. For $\prod_1^1$-sets, we can obtain the

Theorem 8. *Let* $\mathcal{E}$ *be a* $\prod_1^1$*-set which contains a non-void perfect subset. Then there is a non-void perfect subset* P *of* $\mathcal{E}$ *such that*

(i) *it is a* Δ_2^1*-set,*

(ii) *any recursive sieve determining the set* $\mathcal{E}$ *can be bounded on the set* P *by a* Δ_2^1*-ordinal.*

In the proof, we use a result of Tugué-Tanaka [14]. In connection with this theorem, we have the

Theorem 9. *Let* $\mathcal{E}$ *be a* $\prod_1^1$*-set whose measure mes* $(\mathcal{E})$ *is*

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2) Further if we assume E is closed, then E_0 is its perfect kernel. Since E_0 is $\sum_1^1$, it is Π_1^1-closed in the sense of Kreisel [6]. Therefore, when the Baire zero-space is considered, our Theorem 5 implies Theorem 1 of [6].

positive. Then there is a non-empty perfect subset P of $\mathcal{E}$ which belongs to the class Δ_2^1. Consequently, for suitable Δ_2^1-ordinal η_0 mes $(\mathcal{E}_{\eta_0})$ is positive, where $\mathcal{E}_\eta$'s $(\eta<\Omega)$ are the constituents of $\mathcal{E}$ (with respect to any recursive sieve determining $\mathcal{E}$).

This is obtained as a corollary of the preceding theorem by using a result of Kreisel [7]. Corresponding to a theorem of Sélivanowski [12] concerning a CA-set, we have the

Theorem 10. *Let $\mathcal{E}$ be a $\prod_1^1$-set whose measure is positive. Then there is a Δ_2^1-ordinal η_0 such that*

$$\text{mes}(\mathcal{E})=\text{mes}(\bigcup_{\eta<\eta_0}\mathcal{E}_\eta),$$

where $\mathcal{E}_\eta$'s are as in Theorem 9.

§ 4. As is well-known, every analytic set (and hence also CA-set) is measurable. A fortiori so are $\sum_1^1$- and $\prod_1^1$-sets. Kreisel [7] shows that the measure of an arithmetical set in the closed interval [0, 1] is an arithmetical real number. Then what turn out for $\sum_1^1$-sets in N^N? A real number α is said to be a $\sum_1^1(\prod_1^1)$-*real*, if the set of all rationals smaller than α is contained in the class $\sum_1^1(\prod_1^1)$. Then by the method of Kondô-Tugué [5] we can show the

Theorem 11. *The measure of a $\sum_1^1$-set in N^N is a $\sum_1^1$-real, and hence the measure of a $\prod_1^1$-set is a $\prod_1^1$-real. Consequently, the measure of a Δ_1^1-set is a Δ_1^1-real.*

In the near future, we shall publish the detailed proofs of the above theorems elsewhere.

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