102. On Characterizations of I-Algebra. I

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In this paper, we shall show that *an axiomatic system of implicational calculus given by C. A. Meredith is equivalent to Tarski-Bernays' axiom system* using an algebraic formulation.

In his paper [2], Prof. K. Iséki has proved that Tarski-Bernays' axiom system implies Meredith's system and other systems. Further Prof. K. Iséki refers Tarski-Bernays' axiom system as I-algebra. We shall prove that Meredith's alternative 4-axiom set implies Tarski-Bernays' system. We shall carry out this proof algebraically.

Let $\langle X, 0, *\rangle$ be an abstract algebra. For the notion of this algebra and notations, see [1]. The alternative 4-axiom set is given as the following 1–4, $D1$–$D3$.

1 $y*(y*x)\leqslant x$,

2 $(z*x)*(z*y)\leqslant y*x$,

3 $y*x\leqslant(y*x)*x$,

4 $x*(x*y)\leqslant y*(y*x)$,

$D1$ $x\leqslant y$ means $x*y=0$,

$D2$ $0\leqslant x$,

$D3$ $x\leqslant y$, $y\leqslant x$ imply $x=y$.

In 2, put $y*(y*x)$ for y, then we have

$(z*x)*(z*(y*(y*x)))\leqslant(y*(y*x))*x$.

By 1 the right side of the above is equal to 0. Hence, by $D1$, $D2$, and $D3$, we have

5 $z*x\leqslant z*(y*(y*x))$.

If we put $x=(z*y)*x$, $y=(z*x)*(z*(z*y))$, $z=(z*x)*y$ in 2, then we have

$$(((z*x)*y)*((z*y)*x))*(((z*x)*y)*((z*x)*(z*(z*y))))$$
$$\leqslant((z*x)*(z*(z*y)))*((z*y)*x).$$

We see the right side is equal to 0, putting $y=z*y$ in 2. At the same time, we see the second term of the left side is equal to 0, putting $x=y$, $y=z$, $z=z*x$ in 5. Hence we have

6 $(z*x)*y\leqslant(z*y)*x$.

In 2, put $y=y*(y*x)$, $z=x*(x*y)$, and apply 1, 2 to it, we have

7 $(x*(x*y))\leqslant x$.

In 6, put $y=x*y$, $z=x$, then we have $(x*x)*(x*y)\leqslant(x*(x*y))*x$. By 7, the right side is equal to 0. Hence we have

8 $x*x\leqslant x*y$.

In 4, put $y=y*x$, then we have $x*(x*(y*x))\leqslant(y*x)*((y*x)*x)$. By 3, the right side is equal to 0. Hence we have

9 $x\leqslant(x*(y*x))$.

In 8, put $y=x*(y*x)$, then we have $x*x\leqslant x*(x*(y*x))$. By 9, the right side is equal to 0. Hence we have

10 $x\leqslant x$.

In 6, put $z=y$, then we have $(y*x)*y\leqslant(y*y)*x$. The formula 10 means $y*y=0$. Hence we have $0*x=0$ by $D1$ and $D2$. Therefore we have

11 $y*x\leqslant y$.

Theses 2, 9, and 11 are axioms of Tarski-Bernays' system.

Therefore the proof is complete.

References

[1] K. Iséki: Algebraic formulations of propositional calculi. Proc. Japan Acad., **41**, 803-807 (1965).

[2] Y. Imai and K. Iséki: On axiom systems of propositional calculi. XIV. Proc. Japan Acad., **42**, 19-22 (1966).