70. *On Regularity of Solutions of Abstract Differential Equations in Banach Space*

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The present paper is concerned with the estimates for the successive derivatives of solutions of abstract differential equations of parabolic type in a Banach space X:

$$du(t)/dt+A(t)u(t)=f(t), \qquad 0<t\leqq T. \tag{1}$$

The main result is briefly stated as follows: if $A(t)$ and $f(t)$ belong to a Gevrey's class as functions of t, then so does the solution of (1). This is an answer to the problem proposed in p. 388 of [3].

Let $\{M_k\}$ be a sequence of positive numbers which has the properties (1.1), $\cdots$, (1.7) in p. 366 of [4]. In what follows we will not confine ourselves to non quasi-analytic cases since we will not work only in the spaces such as D_{+,M_k} (cf. [3]).

Assumptions. (i) For each $t\in[0, T]$, $A(t)$ is a densely defined linear closed operator in X. The resolvent set of $A(t)$ contains a fixed closed sector $\sum=\{\lambda: \theta\leqq \arg\lambda\leqq 2\pi-\theta\}$, $0<\theta<\pi/2$.

(ii) $A(t)^{-1}$, which is a bounded operator according to the preceding assumption, is infinitely differentiable in t.

(iii) There exist constants K_0 and K such that for any $\lambda\in\sum$, $t\in[0, T]$ and non-negative integer n

$$\|(\partial/\partial t)^n(\lambda-A(t))^{-1}\|\leqq K_0K^nM_n/|\lambda|.$$

It can be shown with the aid of S. Agmon's result on general elliptic boundary value problems ([1]) that the assumptions above are satisfied for the initial-boundary value problems of parabolic differential equations under appropriate conditions on the coefficients.

In view of Theorem 3.1 of [2] the evolution operator $U(t, s)$ can be constructed as follows:

$$U(t, s)=\exp(-(t-s)A(t))+W(t, s),$$
$$W(t, s)=\int_s^t \exp(-(t-\tau)A(t))R(\tau, s)d\tau,$$
$$R(t, s)=\textstyle\sum_{m=1}^{\infty} R_m(t, s),$$
$$R_1(t, s)=-(\partial/\partial t+\partial/\partial s)\exp(-(t-s)A(t)),$$
$$R_m(t, s)=\int_s^t R_1(t, \tau)R_{m-1}(\tau, s)d\tau, \qquad m=2, 3, \cdots.$$

$R(t, s)$ is the solution of the integral equation

$$R(t, s)=R_1(t, s)+\int_s^t R_1(t, \tau)R(\tau, s)d\tau. \tag{2}$$

Theorem 1. *Under the assumptions* (i), (ii), (iii) *there exist constants* L_0, L *such that for any integer* $n\geqq 0$

$$\|(\partial/\partial t)^n U(t,s)\|\leqq L_0 L^n M_n (t-s)^{-n}, \qquad 0\leqq s<t\leqq T.$$

Theorem 2. *Suppose that the assumptions* (i), (ii), (iii) *are satisfied. If* $f(t)$ *is an infinitely differentiable function and satisfies for some constants* B_0 *and* B

$$\|d^n f(t)/dt^n\|\leqq B_0 B^n M_n, \qquad s\leqq t\leqq T,$$

for all integers $n\geqq 0$, *then the solution* $u(t)$ *of* (1) *is infinitely differentiable and satisfies for some constants* N_0 *and* N

$$\|d^n u(t)/dt^n\|\leqq N_0 N^n M_n (t-s)^{-n}, \qquad s<t\leqq T,$$

for all integers $n\geqq 0$.

Lemma 1. *There exist constants* C_0 *and* C *such that*

$$\left\|\left(\frac{\partial}{\partial t}\right)^l\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial s}\right)^m R_1(t,s)\right\|\leqq C_0 C^{m+l} M_l M_m (t-s)^{-l}$$

for all integers $l\geqq 0$ *and* $m\geqq 0$.

Lemma 2. *There exist constants* H_0 *and* H *such that for all integers* $l\geqq 0$

$$\|(\partial/\partial t)^l R(t,s)\|\leqq H_0 H^l M_l (t-s)^{-l}. \tag{3}$$

Outline of proof of Lemma 2. Let us prove the lemma by induction with the respect to l and suppose (3) is true for $l=1,\cdots,n-1$. Let $r_i=s+i(t-s)/(n+1)$, $i=1,\cdots,n$. Then

$$\begin{aligned}(\partial/\partial t)^n R(t,s) &= (\partial/\partial t)^n R_1(t,s)\\ &+\sum_{i=1}^{n}\sum_{j=0}^{i-1}\binom{i-1}{j}\left(\frac{\partial}{\partial t}\right)^{n-i}\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial r_i}\right)^{i-1-j}R_1(t,r_i)\cdot\left(\frac{\partial}{\partial r_i}\right)^{j}R(r_i,s)\\ &+\sum_{i=0}^{n}\int_{r_i}^{r_{i+1}}\sum_{m=0}^{i}\binom{i}{m}\left(\frac{\partial}{\partial t}\right)^{n-i}\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \tau}\right)^{i-m}R_1(t,\tau)\cdot\left(\frac{\partial}{\partial \tau}\right)^{m}R(\tau,s)d\tau.\end{aligned}$$

This can be verified by noting (2) and integrating by part with respect to τ in the right side of

$$\begin{aligned}&\left(\frac{\partial}{\partial t}\right)^n\int_{r_i}^{r_{i+1}}R_1(t,\tau)R(\tau,s)d\tau\\ &=\left(\frac{\partial}{\partial t}\right)^{n-i}\int_{r_i}^{r_{i+1}}\sum_{j=0}^{i}\binom{i}{j}\left(-\frac{\partial}{\partial \tau}\right)^{j}\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \tau}\right)^{i-j}R_1(t,\tau)\cdot R(\tau,s)d\tau.\end{aligned}$$

By the induction hypothesis and with the aid of (1.9), (1.10), (1.10′) in p. 367 of [4] as well as Sterling's formula we get

$$\begin{aligned}\|(\partial/\partial t)^n R(t,s)\|&\leqq \exp(-C_0 M_0^2 eT) H_0 H^n M_n (t-s)^{-n}\\ &+C_0 M_0^2\int_{r_n}^{t}\|(\partial/\partial \tau)^n R(\tau,s)\|d\tau\end{aligned} \tag{4}$$

if H_0 and H are sufficiently large depending only on the constants which appeared in the assumptions (i), (ii), (iii). If we set

$$G(t,s)=(t-s)^n\|(\partial/\partial t)^n R(t,s)\|,$$

then in view of (4) we get

$$G(t, s) \leqq \exp(-C_0 M_0^2 e T) H_0 H^n M_n + C_0 M_0^2 e \int_s^t G(\tau, s) d\tau, \qquad (5)$$

since if $r_n < \tau < t$, $(t-s)^n < (1+n^{-1})^n (\tau - s)^n < e(\tau - s)^n$. Integrating (5) we obtain

$$G(t, s) \leqq H_0 H^n M_n,$$

which completes the proof of the lemma.

The proof of the theorems is similar to the argument above.

References

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