99. *A Note on Extended Regular Functional Spaces*

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1. Beurling and Deny introduced in [1] and [2] the notions of regular functional spaces and Dirichlet spaces. They treatised potentials in such a space. In potential theory, their method is very important for the research of the kernels satisfying the domination principle or the complete maximum principle. But the notion of a regular functional space is not sufficient, because the kernel of a regular functional space is symmetric if it exists. In this note, we shall extend the notion of a regular functional space and show the similar results as Beurling and Deny's. The detail will be published later elsewhere.

2. Let X be a locally compact Hausdorff space where there exists a positive measure ξ satisfying $\xi(\omega)>0$ for any non-empty open set ω in X, and let $C_K=C_K(X)$ be the space of finite continuous functions defined in X with compact support provided with the usual topology.[1] We define an extended regular functional space with respect to X and ξ as follows:

Definition 1. A Banach space $\mathfrak{X}=\mathfrak{X}(X;\xi)$ is called an extended regular functional space (with respect to X and ξ) if each element of $\mathfrak{X}$ is a real-valued locally ξ-summable function defined almost everywhere for ξ simply, *a.e.* in X and the following three conditions are satisfied:

(1.1) For each compact set K in X, there exists a positive constant $A(K)$ such that

$$\int |u(x)|\, d\xi(x) \leq A(K)\, ||u||$$

for any u in $\mathfrak{X}$.

(1.2) The intersection $C_K \cap \mathfrak{X}$ is dense both in C_K and in $\mathfrak{X}$.

(1.3) There exists a continuous bilinear form $\alpha(\cdot,\cdot)$ on $\mathfrak{X}$ such that $\alpha(u,u)=||u||^2$ for any u in $\mathfrak{X}$.

In the above definition, the norm in $\mathfrak{X}$ is denoted by $||u||$. For example, we can construct an extended regular functional space for a uniformly elliptic differential operator of order 2 which is not

1) That is, the net $(f_\alpha)_{\alpha\in I}$ is called to converge f in C_K if there exists a compact set K in X such that the support of f_α is contained in K and (f_α) is uniformly convergent to f.

always self-adjoint (see [9] and [10]). Similarly as in [9] and [10], first we obtain the following

Theorem 1. *Let $\mathfrak{X}$ be an extended regular functional space, and let U be a closed convex subset in $\mathfrak{X}$. For a continuous linear functional F on $\mathfrak{X}$, there exists a unique element $u(F)$ (resp. $\check{u}(F)$) in U such that*

$$\alpha(u(F), v) \geq F(v) \qquad (\textit{resp. } \alpha(v, \check{u}(F)) \geq F(v))$$

for any v in $V_{u(F)}$ (resp. $V_{\check{u}(F)}$), where $V_{u(F)} = \{v \in \mathfrak{X};\ \exists \varepsilon > 0$ such that $\varepsilon v + u(F) \in U\}$ and $V_{\check{u}(F)}$ is similar.

From this theorem, we immediately obtain the following

Corollary 1. *Let $\mathfrak{X}$ be the same as in the above theorem. For any bounded ξ-measurable function f with compact support, there exists a unique element u_f (resp. $\check{u}_f$) in $\mathfrak{X}$ such that*

$$\alpha(u_f, v) = \int v(x) f(x) d\xi(x) \qquad (\textit{resp. } \alpha(v, \check{u}_f) = \int v(x) f(x)\, d\xi(x))$$

for any v in $\mathfrak{X}$.

The element u_f (resp. $\check{u}_f$) is called the potential (resp. the adjoint potential) generated by f. More generally, we define a potential in $\mathfrak{X}$ as follows:

Definition 2. Let $\mathfrak{X}$ be an extended regular functional space. For an element u in $\mathfrak{X}$, if there exists a real Radon measure μ in X satisfying

$$\alpha(u, f) = \int f d\mu \qquad (\textit{resp. } \alpha(f, u) = \int f\, d\mu)$$

for any f in $C_K \cap \mathfrak{X}$, we call this element u the potential (resp. the adjoint potential) generated by μ, and denote it by u_μ (resp. $\check{u}_\mu$). Especially if μ is positive, u_μ (resp. $\check{u}_\mu$) is called the pure potential (resp. the adjoint pure potential) generated by μ.

Evidently we obtain the following

Remark 1. *For a real Radon measure μ in X, the potential u_μ generated by μ exists in $\mathfrak{X}$ if and only if the adjoint potential $\check{u}_\mu$ generated by μ exists in $\mathfrak{X}$. Furthermore the equality $\alpha(u_\mu, u) = \alpha(u, \check{u}_\mu)$ holds for any u in $\mathfrak{X}$.*

3. We shall consider the resolvent operator associated with an extended regular functional space. Similarly as in [2], we obtain the following

Theorem 3. *Let $\mathfrak{X}$ be an extended regular functional space, and let λ be a given positive number. For any u in $\mathfrak{X}$ or $L^2 = L^2(\xi)$, there exists a unique element $R_\lambda u$ (resp. $\check{R}_\lambda u$) in $\mathfrak{X}$ such that*

$$\alpha(R_\lambda u, v) = \int (R_\lambda u - u) v d\xi \qquad (\textit{resp. } \alpha(v, \check{R}_\lambda u) = \int (\check{R}_\lambda u - u) v d\xi)$$

for any v in $L^2 \cap \mathfrak{X}$.

The operator $R_\lambda, \check{R}_\lambda \colon \mathfrak{X} \to \mathfrak{X}$, or $L^2 \to L^2$ is bounded and $R_\lambda \to I$ and

$R_\lambda \to I$ weakly both in $\mathfrak{X}$ and in L^2, respectively, as $\lambda \to 0$, where I is the identity operator. These operators R_λ and $\check{R}_\lambda$ are called the resolvent operator and the adjoint resolvent operator, respectively. Then the equality $\alpha(R_\lambda u, v) = \alpha(u, \check{R}_\lambda v)$ holds for any u, v in $\mathfrak{X}$ and any $\lambda > 0$.

4. Now we consider the domination principle. First we give some definitions.

Definition 3. Let $\mathfrak{X}$ be an extended regular functional space. We say that the left-module contraction (resp. the right-module contraction) operates in $\mathfrak{X}$ if, for any u in $\mathfrak{X}$, the function $|u|$ is contained in $\mathfrak{X}$ and the inequality $\alpha(u + |u|, u - |u|) \geq 0$ (resp. $\alpha(u - |u|, u + |u|) \geq 0$) holds.

Evidently we obtain that the left-module contraction operates in $\mathfrak{X}$ il and only if the right-module contraction operates in $\mathfrak{X}$.

Definition 4. Let $\mathfrak{X}$ be an extended regular functional space. We say that $\mathfrak{X}$ satisfies the left (resp. the right) domination principle if, for a positive bounded ξ-measurable function f with compact support and a pure potential u_μ (resp. an adjoint pure potential $\check{u}_\mu$) in $\mathfrak{X}$, the condition that $u_f(x) \leq u_\mu(x)$ (resp. $\check{u}_f(x) \leq \check{u}_\mu(x)$) holds *a.e.*[2] in the set $\{x \in X; f(x) > 0\}$ implies that the same inequality holds *a.e.* in X.

The first main theorem is following

Theorem 3. *Let $\mathfrak{X}$ be an extended regular functional space. Then the following six conditions are equivalent.*

(A_1) *The left-module contraction operates in $\mathfrak{X}$.*

(A_2) *$\mathfrak{X}$ satisfies the left domination principle.*

(A_3) *For any $\lambda > 0$, the resolvent operator R_λ is non-negative, i.e., $R_\lambda f \geq 0$ for any f in $\mathfrak{X} \cup L^2$ such that $f \geq 0$.*

(A_4) *The right-module contraction operates in $\mathfrak{X}$.*

(A_5) *$\mathfrak{X}$ satisfies the right domination principle.*

(A_6) *For any $\lambda > 0$, the adjoint resolvent operator $\check{R}_\lambda$ is non-negative.*

In order to prove the implications $(A_1) \Rightarrow (A_2)$ and $(A_4) \Rightarrow (A_5)$, the following lemma is essential.

Lemma 1. *Let $\mathfrak{X}$ be an extended regular functional space in which the left-module contraction operates. For two pure potentials u_{μ_1} and u_{μ_2} in $\mathfrak{X}$, there exists a pure potential u_μ in $\mathfrak{X}$ such that*

$$u_\mu = \inf(u_{\mu_1}, u_{\mu_2}) .$$

The proof of the implications $(A_2) \Rightarrow (A_3)$ and $(A_5) \Rightarrow (A_6)$ is followed from Theorem 2 and the following lemma.

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2) A property is said to hold *a.e.* in a ξ-measurable set E if the property holds locally almost everywhere for ξ in E.

Lemma 2. *Let $\mathfrak{X}$ be an extended functional space. For a locally ξ-summable function f in X, suppose that the potential u_f generated by f exists in $\mathfrak{X}$. Then there exist a sequence (f_n) of non-negative bounded ξ-measurable functions with compact support and a sequence (u_{μ_n}) of pure potentials in $\mathfrak{X}$ such that $f_n \leq f^+$, $u_{f_n} - u_{\mu_n} \leq u_f$ and the sequence $(u_{f_n} - u_{\mu_n})$ converges weakly to u_f in $\mathfrak{X}$ as $n \longrightarrow +\infty$.*

Finally we can prove the implications $(A_3) \Rightarrow (A_1)$ and $(A_6) \Rightarrow (A_4)$ by using the following two lemmas.

Lemma 3. *Let $\mathfrak{X}$ be an extended regular functional space which satisfies the condition (A_3). For any $\lambda > 0$, there exists a positive measure σ_λ in the product space $X \times X$* such that

$$\int R_\lambda f(x) g(x) d\xi(x) = \iint f(x) g(y) d\sigma_\lambda(x, y)$$

for any f, g in C_K.

Lemma 4. *Let $\mathfrak{X}$ be an extended regular functional space. For a function f in C_K, suppose that*

$$H_\lambda(f) = \frac{1}{\lambda}\left(\int |f|^2 d\xi - \iint f(x) f(y) d\sigma_\lambda(x, y)\right)$$

is bounded for λ. Then f is contained in $\mathfrak{X}$.

In a functional space with an additional condition, Deny [4] proved the similar equivalence as one between (A_1) and (A_2). Kishi [8] proved the similar equivalence as one between (A_2) and (A_4) in the potential theory with respect to a continuous non-symmetric kernel.

5. In this section, we consider the complete maximum principle.

Definition 5. Let $\mathfrak{X}$ be an extended regular functional space. We say that $\mathfrak{X}$ satisfies the left (resp. the right) complete maximum principle if, for a positive bounded ξ-measurable function f with compact support and a pure potential u_μ (resp. an adjoint pure potential $\check{u}_\mu$) in $\mathfrak{X}$, the condition that $u_f(x) \leq u_\mu(x) + 1$ (resp. $\check{u}_f(x) \leq \check{u}_\mu(x) + 1$) holds *a.e.* in the set $\{x \in X; f(x) > 0\}$ implies that the same inequality holds *a.e.* in X.

Then we obtain the second main theorem.

Theorem 4. *Let $\mathfrak{X}$ be an extended regular functional space. Then $\mathfrak{X}$ satisfies the left complete maximum principle if and only if, for any $\lambda > 0$, the resolvent operator R_λ is sub-markovian, i.e., $0 \leq R_\lambda f \leq 1$ for any f in $\mathfrak{X} \cup L^2$ such that $0 \leq f \leq 1$.*

Using Lemma 2, we can prove the "only if" par in the same method as Deny [4]. For the proof of the "if" part, the following lemma is essential.

Lemma 5. *Let $\mathfrak{X}$ be an extended regular functional space.*

Suppose that the resolvent operator R_λ is sub-markovian for any $\lambda>0$. For an adjoint potential $\check{u}_\mu$ in $\mathfrak{X}$, if $\check{u}_\mu\geq 0$ a.e. in X, $\int d\mu\geq 0$.

Now we consider the unit contraction. The projection T on the real line R into the closed interval $[0, 1]$ is called the unit contraction.

Definition 6. We say that the left-unit (resp. the right-unit) contraction operates in $\mathfrak{X}$ if, for any u in $\mathfrak{X}$, the function $T\cdot u$ is in $\mathfrak{X}$ and the inequality

$$\alpha(u+T\cdot u, u-T\cdot u)\geq 0 \qquad (\text{resp. } \alpha(u-T\cdot u, u+T\cdot u)\geq 0)$$

holds.

Remark 2. *In the case that $\mathfrak{X}$ is a regular functional space, the above definition is equal to Deny's (Cf. [4]).*

Finally we obtain the following third main theorem.

Theorem 5. *Let $\mathfrak{X}$ be an extended regular functional space. Then the following three conditions are equivalent.*

(B_1) $\mathfrak{X}$ satisfies the left (resp. the right) complete maximum principle.

(B_2) For any $\lambda>0$, the resolvent operator R_λ (resp. the adjoint resolvent operator $\check{R}_\lambda$) is sub-markovian.

(B_3) The left-unit (resp. the right-unit) contraction operates in $\mathfrak{X}$.

By Theorem 4, the equivalence between (B_1) and (B_2) is known. In order to show the implication (B_2)$\Rightarrow$(B_3), we need the following

Lemma 6. *Let $\mathfrak{X}$ be an extended regular functional space which satisfies the condition (B_2). Then for any $\lambda>0$,*

$$\sigma_\lambda(X\times A)\leq\xi(A) \qquad (\textit{resp. } \sigma_\lambda(A\times X)\leq\xi(A))$$

for any Borel set A in X.

The implication (B_3)$\Rightarrow$(B_1) can be proved in the same method as in the case of a Dirichlet space (see [2] and [4]).

Remark 3. *The relation between the left complete maximum principle and the right complete maximum principle is unknown.*

Definition 7. An extended regular functional space is called an extended Dirichlet space if the left- and the right-unit contraction operate in such a space.

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