138. *Closures and Neighborhoods in Certain Proximity Spaces*

By Eiichi HAYASHI

Nagoya Institute of Technology

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In the paper [1], we defined a binary relation δ, called a paraproximity, for a pair of subsets of a point set R. We there proved that a paraproximity yields a completely normal space ([1] Theorem 1). Further we showed that, for a pair of subsets A and B of a paraproximity space R, $(A, B) \in \delta$ implies $A \cap \bar{B} \neq \emptyset$ ([1] Theorem 2). In the present paper we show that the converse of Theorem 2 holds. Hence a paraproximity structure which is compatible with the topology is uniquely determined. The remaining parts of this paper are devoted to the study of the neighborhood.

First we restate the definition of a paraproximity. By a *paraproximity* on a set R we mean a binary relation δ for pairs of subsets of R satisfying the following axioms:

Axiom (1). For every $A \subset R$, $(A, \emptyset) \notin \delta$, and $(\emptyset, A) \notin \delta$. (We add the latter condition $(\emptyset, A) \notin \delta$ to Axiom (1) of [1].)

Axiom (2). $(A, B \cup C) \in \delta$ if and only if either $(A, B) \in \delta$ or $(A, C) \in \delta$.

Axiom (3). For an arbitrary index set Λ, $(\bigcup_{\lambda \in \Lambda} A_\lambda, B) \in \delta$ if and only if there exists an index $\mu \in \Lambda$ satisfying the relation $(A_\mu, B) \in \delta$.

Axiom (4). For arbitrary two points $a, b \in R$, $(\{a\}, \{b\}) \in \delta$ if and only if $a = b$.

Axiom (5). If $(A, B) \notin \delta$ and $(B, A) \notin \delta$, then there exist two disjoint subsets U and V satisfying:

$$(A, R-U) \notin \delta, \qquad (U, R-U) \notin \delta:$$
$$(B, R-V) \notin \delta, \qquad (V, R-V) \notin \delta.$$

We note that the next lemma (Steiner [3]) follows from Axiom (3).

Lemma 1. *$(A, B) \in \delta$ if and only if $(\{x\}, B) \in \delta$ for some x in A.*

Lemma 2. *If $(\{x\}, A) \notin \delta$ then $(A, \{x\}) \notin \delta$.*

Proof. *If* $(\{x\}, A) \notin \delta$, then $x \notin A$ by [1, Lemma 3]. Suppose that $(A, \{x\}) \in \delta$. Then, by Lemma 1, there is a point a in A such that $(\{a\}, \{x\}) \in \delta$. From Axiom (4) follows $a = x$ which is a contradiction.

1. Let R be a set with a paraproximity δ. *A* set $B \subset R$ is said to be a *paraproximal neighborhood* of a set $A \subset R$ (notation:

$A \Subset B$) if and only if $(A, R-B) \notin \delta$. (cf. Y. M. Smirnov [2].)

Lemma 3. *The relation* $\Subset$ *satisfies the following conditions*:

(1) $R \Subset R$, $\emptyset \Subset \emptyset$.

(2) $A \subset B \Subset C \subset D$ *implies* $A \Subset D$.

(3) $A \Subset B_i$, $i=1, 2, \cdots, n$, *if and only if* $A \Subset \bigcap_{i=1}^n B_i$.

(4) *For any index set* Λ, $\bigcup_{\lambda \in \Lambda} A_\lambda \Subset B$ *if and only if* $A_\lambda \Subset B$ *for every* λ.

(5) $x \Subset R-y$ *if and only if* $x \neq y$.

(6) *If* $A \Subset R-B$ *and* $B \Subset R-A$ *then there are two disjoint subsets* U *and* V *such that* $U \Subset U$, $V \Subset V$, $A \Subset U$, *and* $B \Subset V$.

Proof. (1) follows from Axiom (1). To prove (2), assume that $A \subset B \Subset C \subset D$. Then $(B, R-C) \notin \delta$ and hence from [1, Lemma 1 and 2] $(A, R-D) \notin \delta$, which implies $A \Subset D$. (3) is established by repeated application of Axiom (2). If $\bigcup_{\lambda \in \Lambda} A_\lambda \Subset B$, then by (2), $A_\lambda \Subset B$ for every λ. The converse implication of (4) is obvious from Axiom (3), (5), and (6) evidently follow from Axiom (4) and (5) respectively.

Theorem 1. *Let a relation* $\Subset$ *satisfying the conditions* (1)—(6) *of Lemma* 3, *be defined on the family of subsets of* R. *Then* δ, *defined by* $(A, B) \in \delta$ *if and only is* $A \not\Subset R-B$, *is a paraproximity on* R.

Proof. We must show that δ as defined in terms of the relation $\Subset$, satisfies Axiom (1)—(5). For every $A \subset R$, we have from Lemma 3 (1) and (2), that $A \Subset R$ and hence $(A, \emptyset) \notin \delta$. Similarly we have that $(\emptyset, A) \notin \delta$. Thus Axiom (1) holds. To prove Axiom (2), let $(A, B \cup C) \in \delta$. Then $A \not\Subset R-(B \cup C) = (R-B) \cap (R-C)$. Assume, on the contrary, that $A \Subset R-B$ and $A \Subset R-C$. Then from Lemma 3 (3), we have that $A \Subset (R-B) \cap (R-C)$, which is impossible. Therefore $A \not\Subset R-B$ or $A \not\Subset R-C$. This proves "only if" part of Axiom (2). "if" part of Axiom (2) also follows from (3) in Lemma 3. Axioms (3), (4), and (5) are derived from (4), (5), and (6) in Lemma 3 respectively.

Moreover, paraproximal neighborhoods have the following properties:

Theorem 2.

(1) $A \Subset B$ *implies* $A \subset B$.

(2) *If* $A_i \Subset B_i$, $i=1, 2, \cdots n$, *then* $\bigcap_{i=1}^n A_i \Subset \bigcap_{i=1}^n B_i$ *and* $\bigcup_{i=1}^n A_i \Subset \bigcup_{i=1}^n B_i$.

(3) *Let* A *be a subset of* R *and* x *a point of* R. *Then* $x \Subset R-A$ *implies* $A \Subset R-x$.

(4) *For every point* x, $R-x \Subset R-x$.

(5) $A \Subset B$ *implies the existence of a set* $U \subset R$ *such that* $A \Subset U \Subset B$ *and* $U \Subset U$.

(6) $A \Subset B$ *implies the existence of sets* U *and* $V \subset R$ *such that*

$A \Subset U \Subset V \Subset B$, $U \Subset U$, *and* $V \Subset V$.

Proof. To prove (1), let $A \Subset B$. By definition, it follows that $(A, R-B) \notin \delta$ and so $A \cap (R-B) = \emptyset$ from [1, Lemma 3]. We show that (2) holds in the case of $n=2$. Let $A_i \Subset B_i$, that is, $(A_i, R-B_i) \notin \delta$, $i=1, 2$. Then by [1, Lemma 2], we have $(A_1 \cap A_2, R-B_i) \notin \delta (i=1, 2)$ and so $(A_1 \cap A_2, R-(B_1 \cap B_2)) \notin \delta$ from Axiom (2). It follows that $A_1 \cap A_2 \Subset B_1 \cap B_2$. Assume again that $A_i \Subset B_i (i=1, 2)$. Since $(A_i, R-B_i) \notin \delta (i=1, 2)$, we have $(A_1, R-(B_1 \cup B_2)) \notin \delta$ and $(A_2, R-(B_1 \cup B_2)) \notin \delta$ and hence $A_1 \cup A_2 \Subset B_1 \cup B_2$. (3) and (4) are immediate consequences of Lemma 2 and [1, Lemma 4], respectively.

To establish (5), assume that $A \Subset B$, that is, $(A, R-B) \notin \delta$. From [1, Lemma 2], $(\{x\}, R-B) \notin \delta$ for every x in A and hence $(R-B, \{x\}) \notin \delta$ by Lemma 2. On account of Axiom (5), there exist two disjoint subsets U_x and V_x ($x \in$ A) such that $x \Subset U_x$, $U_x \Subset U_x$, $R-B \Subset V_x$, and $V_x \Subset V_x$. Put $U = \bigcup_{x \in A} U_x$. From Lemma 3(2) we have $x \Subset U$ since $x \Subset U_x \subset U$ for every $x \in A$. Hence by Lemma 1, $(A, R-U) \notin \delta$ so that $A \Subset U$. Since $U_x \Subset U_x \subset U$, from Lemma 3(2) we have $U_x \Subset U$ for every $x \in A$. By Lemma 3(4), $\bigcup_{x \in A} U_x \Subset U$ and hence $U \Subset U$. Further since $U_x \cap V_x = \emptyset$ and $R-B \subset R-U_x$ for every $x \in A$, we have $U \subset B$. Since $U \Subset U \subset B$, it follows from Lemma 3 (2) that $U \Subset B$. Consequently U is the required set. (6) follows from a twofold application of (5).

3. Let R be a set and δ a paraproximity on R. For $A \subset R$, we set $(*)$ $c(A) = \{x : x \in R, (\{x\}, A) \in \delta\}$.

Lemma 4. *For* $U \subset R$, $U \Subset U$ *if and only if* $c(R-U) = R-U$.

Proof. Suppose that $U \Subset U$ i.e. $(U, R-U) \notin \delta$. Then it follows from [1, Lemma 2] that for every $x \in U$, $(\{x\}, R-U) \notin \delta$ and so $x \notin c(R-U)$. Therefore we have $c(R-U) \subset R-U$. For every $x \in R-U$, we have $(\{x\}, R-U) \in \delta$ by [1, Lemma 3] and hence $x \in c(R-U)$. Consequently $c(R-U) = R-U$.

To show the converse, assume that $c(R-U) = R-U$ and that $U \not\Subset U$. Then by Lemma 1, there is an $x \in U$ such that $(\{x\}, R-U) \in \delta$. Hence $x \in c(R-U) = R-U$, which is impossible. This completes the proof of Lemma.

Lemma 5. *The operator c defined by $(*)$ satisfies the Kuratowski closure axioms.*

Proof. (1) $c(\emptyset) = \emptyset$: Assume $x \in c(\emptyset)$, so that $(\{x\}, \emptyset) \in \delta$. This is impossible on account of Axiom (1).

(2) $A \subset B$ implies $c(A) \subset c(B)$: This is an immediate consequence of [1, Lemma 1].

(3) $c(A \cup B) \subset c(A) \cup c(B)$ follows from Axiom (2).

(4) $A \subset c(A)$: If x is an arbitrary point of A, then $c(x) \subset c(A)$ by (2). On account of Axiom (4), we have $(\{x\}, \{x\}) \in \delta$ and hence

$x \in c(x)$, from which it follows that $x \in c(A)$ and that $A \subset c(A)$.

(5) $cc(A) \subset c(A)$: We assume that $x \notin c(A)$. Then $(\{x\}, A) \notin \delta$ implies $(A, \{x\}) \notin \delta$ from Lemma 2. By Axiom (5), we can find two disjoint sets U and V such that $A \Subset U$, $U \Subset U$, $x \Subset V$, and $V \Subset V$. It follows from (2) and Lemma 4 that $c(A) \subset c(R-V) = R-V$. Using again (2) and Lemma 4, we have $cc(A) \subset R - V \not\ni x$. This ends the proof of (5).

(6) $x = c(x)$ follows from Axiom (4).

The operator c is called a *paraproximal closure operator* (or simply *closure operator*). The set R topologized by Lemma 5 is called a *paraproximity space* (R, δ, c). Lemma 4 implies that $U \subset R$ is open if and only if $U \Subset U$.

Theorem 3. *Let (R, δ, c) be a paraproximity space. For A, $B \subset R$, the following statements are equivalent.*

(1) $(A, B) \in \delta$.

(2) $A \cap c(B) \neq \emptyset$.

(3) $A \not\Subset R-B$.

Proof. The equivalence of (1) and (3) follows from our definition. It suffices to show the equivalence of (1) and (2). If $(A, B) \in \delta$ then by Lemma 1 there exists a point x in A such that $(\{x\}, B) \in \delta$. Hence $A \cap c(B) \neq \emptyset$. Conversely, assume that $x \in A \cap c(B)$. Then $(\{x\}, B) \in \delta$ and so $(A, B) \in \delta$ from [1, Lemma 2].

Corollary. *$A \Subset B$ implies $A \Subset$ int $B \Subset B$. (int B denotes the interior of B.)*

Proof. Since int $B = R - c(R-B)$ by definition, we have $\operatorname{int} B \cap c(R-B) = \emptyset$. It follows from Theorem 3 that $\operatorname{int} B \Subset B$. To prove that $A \Subset \operatorname{int} B$, assume $A \Subset B$. Since $A \cap c(R-B) = \emptyset$ and $c(R-B) = R - \operatorname{int} B$, we have that $A \cap c(R - \operatorname{int} B) = A \cap (R - \operatorname{int} B) = \emptyset$. By Theorem 3, this implies $A \Subset \operatorname{int} B$.

By using Theorem 3 we can simplify the proof of the following [1, Theorem 1].

Theorem 4. *The paraproximity space (R, δ, c) is completely normal.*

Proof. As previously showed (Lemma 5), R is a T_1-space. It suffices to prove that R satisfies the T_5 axiom of separation. Let A and B be separated in R (i.e. $A \cap c(B) = \emptyset$ and $B \cap c(A) = \emptyset$). By Theorem 3, this implies $(A, B) \notin \delta$ and $(B, A) \notin \delta$. Therefore by virtue of Axiom (5), there exist disjoint open sets U and V such that $A \Subset U$ and $B \Subset V$.

References

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