6. *On Zero Entropy and Quasi-discrete Spectrum for Automorphisms*

By Nobuo AOKI

Department of Mathematics, Josai University, Sakado, Saitama

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§ 1. Abramov [1] has defined the notion of an automorphism with quasi-discrete spectrum. Hahn and Parry [7] have developed an analogous theory for homeomorphisms of compact spaces, and Parry [10] has shown that the maximal partition of an ergodic affine transformation of a compact connected metric abelian group and that of the ergodic affine transformation with quasi-discrete spectrum coincide. In §3 we prove that totally ergodic automorphisms belonging to $C_2(T)$ [3] have quasi-discrete spectrum if and only if the automorphisms have zero entropy. The study in this paper depends on [4], [10], and [16].

§ 2. Let (X, Σ, m) be a Lebesgue measure space with normalized measure m. We denote by $\Sigma(m)$ the Boolean σ-algebra by identifying sets in Σ whose symmetric difference has zero measure, and the measure m is induced on the elements of $\Sigma(m)$ in the natural way. Let $L^2(\Sigma)$ be the Hilbert space of complex-valued square integrable functions defined on (X, Σ, m) and let $L^\infty(\Sigma)$ be the Banach space of complex-valued m essentially bounded functions defined on (X, Σ, m) but sometimes we use $L^2(\Sigma(m))$ instead of $L^2(\Sigma)$. Let T be automorphism of (X, Σ, m) and we denote by $V_T: f(x)\to f(Tx)$ $(x\in X,\ f\in L^2(\Sigma))$ the linear isometry induced by T. T is said to be *totally ergodic* if T^n is ergodic for every positive integer n and to be a *Kolmogorov automorphism* if there exists sub σ-field $\mathcal{B}$ such that (1) $\mathcal{B}\subset T^{-1}\mathcal{B}$ (2) $\bigcap_{n=-\infty}^{\infty} T^n\mathcal{B}=\mathcal{Q}$ ($\mathcal{Q}$ a field whose measurable sets are measure zero or one) and (3) $\bigvee_{n=-\infty}^{\infty} T^n\mathcal{B}=\Sigma$. If there is a basis $\boldsymbol{O}$ of $L^2(\Sigma)$ each term of which is a normalized proper function of T, then T is said to have *discrete spectrum*. Clearly $\boldsymbol{O}$ includs a circle group K. If T is ergodic then it turns out that $|f|=1$ a.e. for each $f\in\boldsymbol{O}$, and that $\boldsymbol{O}=K\times\boldsymbol{O}(T)$ where $\boldsymbol{O}(T)$ is a subgroup of $\boldsymbol{O}$ isomorphic to the factor group $\boldsymbol{O}/K$. If T is totally ergodic and has discrete spectrum, then $C_1(T)\neq C_2(T)=C_3(T)$ [3]. If T is ergodic and has discrete spectrum, then for every $Q\in C_2(T)$ there exist almost automorphisms W, S such that W has each function of $\boldsymbol{O}(T)$ as a proper function and V_S maps $\boldsymbol{O}(T)$ onto itself, and $Q=WS$ a.e. [3] and [4]. Let T be ergodic, then for an automorphism S satisfying $V_S\boldsymbol{O}(T)=\boldsymbol{O}(T)$

we denote by $B(S)$ the homomorphism on $\boldsymbol{O}(T)$, $B(S)f=f^{-1}V_S f$. We put $\boldsymbol{O}_S(T)_n=\{f\in \boldsymbol{O}(T): B(S)^n f=1 \text{ a.e.}\}$, $n=1, 2, \cdots$, then it turns out that $\boldsymbol{O}_S(T)_1\subset \boldsymbol{O}_S(T)_2\subset\cdots$, and that $\boldsymbol{O}_S(T)_n$ is a subgroup. Let Q be a totally ergodic automorphism of (X, Σ, m), we recall the following definition of quasi-proper functions [1]. Let $G(Q)_0$ be a set $\{\alpha\in K: V_Q f=\alpha f \text{ a.e.}, \|f\|_2=1 \text{ for } f\in L^2(\Sigma)\}$. For $i>0$ let $G(Q)_i\subset L^2(\Sigma)$ be the set of all normalized functions f such that $V_Q f=gf$ a.e. where $g\in G(Q)_{i-1}$. The set $G(Q)_i$ is the set of quasi-proper functions of order i. We put $G(Q)=\bigcup_i G(Q)_i$. It turns out that $|f|=1$ a.e. for each $f\in G(Q)$, and that $G(Q)=K\times \boldsymbol{O}(Q)$ where $\boldsymbol{O}(Q)$ is a subgroup of $G(Q)$. Q is said to have *quasi-discrete spectrum* if $G(Q)$ spans $L^2(\Sigma)$. The definition according to the improved version of entropy is given by Sinai [14] as following: for any finite subfield $\mathcal{J}$ of Σ denote the *entropy* $H(\mathcal{J})$ of $\mathcal{J}$ by $H(\mathcal{J})=-\Sigma_k m(A_k)\log m(A_k)$ where the sum is taken over the finite atoms A_k of $\mathcal{J}$ and the entropy $h(T, \mathcal{J})$ of an automorphism T with respect to a finite subfield $\mathcal{J}$ is defined by $h(T, \mathcal{J})=\lim_{n\to\infty}\frac{1}{n}H\left(\bigvee_{j=1}^{n-1}T^{-j}\mathcal{J}\right)$, and the *entropy* $h(T)$ of T is defined as $h(T)=\sup\{h(T, \mathcal{J}): \mathcal{J} \text{ finite}, \mathcal{J}\subset\Sigma\}$. We can consider T restricted to a T-invariant sub σ-field $\mathcal{B}$ and obtain a corresponding entropy $h_{\mathcal{B}}(T)=\sup\{h(T, \mathcal{J}): \mathcal{J} \text{ finite}, \mathcal{J}\subset\mathcal{B}\}$. It is known that T has completely positive entropy if and only if T is a Kolmogorov automorphism [13]. A necessary and sufficient condition that a closed subspace M of $L^2(\Sigma)$ be of the form $M=L^2(\mathcal{C}(m))$ where $\mathcal{C}(m)$ is the smallest σ-algebra of $\Sigma(m)$ with respect to which all functions in M are measurable is that M contain a dense subalgebra consisting of bounded functions, constant functions and their complex conjugations [5]. If β is an ergodic automorphism on a compact abelian group, then β is a Kolmogorov automorphism [12].

§3. Throughout we consider an ergodic automorphism T of (X, Σ, m) having discrete spectrum.

Proposition 1. *Let Q be a totally ergodic automorphism. If Q has quasi-discrete spectrum, then there exist almost automorphisms W, S such that W has each function of $\boldsymbol{O}(Q)$ as a proper function and V_S maps $\boldsymbol{O}(Q)$ onto itself and $Q=WS$* a.e.

Proof. Since Q is a totally ergodic automorphism having quasi-discrete spectrum, $\boldsymbol{O}(Q)$ is an orthonormal base of $L^2(\Sigma)$ [1]. V_Q is an automorphism $G(Q)$ onto itself and a subgroup $K\times 1$ is mapped identically onto itself. We define maps $P:\boldsymbol{O}(Q)\to\boldsymbol{O}(Q)$ and $R:\boldsymbol{O}(Q)\to K$ by $V_Q f=Rf\cdot Pf$ for $f\in\boldsymbol{O}(Q)$. Since V_Q is an automorphism $G(Q)$ onto itself, $R(f_1f_2)=Rf_1\cdot Rf_2$, $P(f_1f_2)=Pf_1Pf_2$ a.e. for $f_1, f_2\in\boldsymbol{O}(Q)$. It turns out that P is an automorphism of $\boldsymbol{O}(Q)$. To define the linear isometry, we put $V\left(\sum_{k=1}^{n} r_k f_k\right)=\sum_{k=1}^{n} r_k P f_k (f_k \in \boldsymbol{O}(Q))$. Then V is an isometry which can be extended uniquely to an isometry of $L^2(\Sigma)$ onto itself, and we suppose that V is a linear isometry of $L^2(\Sigma)$ onto itself. The proof of $VL^\infty(\Sigma)=L^\infty(\Sigma)$ and multiplication of V restricted to $L^\infty(\Sigma)$ is similar to a proof in [4]. By multiplication theorem there exists an almost automorphism S such that $V=V_S$ on $L^2(\Sigma)$. Furthermore we define a map $V': \boldsymbol{O}(Q) \to \{Rf\cdot f : f \in \boldsymbol{O}(Q)\}$ by $V'f=Rf\cdot f$. Then V' has a unique continuous extension V'' on $L^2(\Sigma)$ such that $V'f=V''f$ for each $f \in \boldsymbol{O}(Q)$. By the above way there exists an almost automorphism W such that $V''=V_W$ on $L^2(\Sigma)$. Let $f \in \boldsymbol{O}(Q)$, then $V_Q f=V_S V_W f$ a.e. Therefore we can conclude that $Q=WS$ a.e.

Proposition 2. *Let S be an automorphism satisfying $V_S\boldsymbol{O}(T)=\boldsymbol{O}(T)$ and let W be an automorphism which has each function of $\boldsymbol{O}(T)$ as a proper function. If WS is a totally ergodic automorphism and $\boldsymbol{O}(T)=\bigcup_{n=1}^{\infty}\boldsymbol{O}_S(T)_n$, then WS has quasi-discrete spectrum.*

Proof. We put $Q=WS$. For any $f \in \boldsymbol{O}(T)$, there exists an integer n such that $f \in \boldsymbol{O}_S(T)_n$. We show by induction that if n is the least integer for which $f \in \boldsymbol{O}_S(T)_n$ then f is a proper function of $G(Q)_n$. If $f \in \boldsymbol{O}_S(T)_1$ then $V_Q f=\alpha f$ a.e. Therefore f is a proper function of Q. Suppose now that every $f \in \boldsymbol{O}_S(T)_n$ is a quasi-proper function of $G(Q)_n$. Let f be a function of $\boldsymbol{O}_S(T)_{n+1}$, then $V_Q f=\alpha B(S)f\cdot f$ a.e. and $\alpha B(S)f$ is a quasi-proper function of $G(Q)_n$, by the inductive hypothesis and the fact $B(S)^{n+1}f=B(S)^n(B(S)f)=1$ a.e. Therefore the function f is a quasi-proper function of $G(Q)_{n+1}$. Since $\boldsymbol{O}(T)$ is an orthonormal base of $L^2(\Sigma)$, we see that Q has quasi-discrete spectrum.

Proposition 3. *Let S be an automorphism such that $V_S\boldsymbol{O}(T)=\boldsymbol{O}(T)$ and let W be an automorphism which has each function of $\boldsymbol{O}(T)$ as a proper function. If S is ergodic on $(X, \mathcal{C}, m)$ where $\mathcal{C}$ is a non-trivial S-invariant sub σ-field of Σ* and if $O(T)'$ is a subgroup of $O(T)$ such that $L^2(\mathcal{C})=\overline{\text{span } O(T)'}$, *then $h(WS)>0$.*

Proof. Since $\boldsymbol{O}(T)'$ is a subgroup of $\boldsymbol{O}(T)$, we denote by X' the character group of the discrete abelian group $\boldsymbol{O}(T)'$. Then X' is a compact metric abelian group with normalized complete Haar measure. Let $\langle\cdot,\cdot\rangle$ denote the pairing between X' and its dual $\boldsymbol{O}(T)'$. To define the linear isometry we put $V\left(\sum_{k=1}^{n} r_k f_k\right)=\sum_{k=1}^{n} r_k\langle\cdot, f_k\rangle$ $(f_k \in \boldsymbol{O}(T)')$. Then V is an isometry which can be extended uniquely to an isometry of $L^2(\mathcal{C})$ onto $L^2(B)$ (B a complete Borel class of X'). We suppose that V is a linear isometry of $L^2(\mathcal{C})$ onto $L^2(B)$. We observe that $VL^\infty(\mathcal{C})=L^\infty(B)$ and $V(fg)=VfVg$ a.e. for $f, g \in \boldsymbol{O}(T)'$. Therefore, by multiplication theorem there exists an isomorphism φ such that $V=V_\varphi$.

Now define V', V'' on $L^2(B)$ by $V'=V_\varphi V_S V_\varphi^{-1}$, $V''=V_\varphi V_W V_\varphi^{-1}$ respectively. By Pontrjagin's duality theorem there exist a continuous group automorphism β on X' such that $V'=V_\beta$ on $L^2(B)$, and a rotation ξ such that $V'=V_\xi$ on $L^2(B)$. Since S is isomorphic to β and S is ergodic on $(X, \mathcal{C}, m)$, β is ergodic. Therefore β is a Kolmogorov automorphism and $\xi\beta$ has completely positive entropy. Therefore we have $h_{\mathcal{C}}(WS) = h(\xi\beta) > 0$.

Corollary 1. *Let S be an automorphism such that $V_S\boldsymbol{O}(T)=\boldsymbol{O}(T)$. If S is ergodic, then S is a Kolmogorov automorphism.*

The proof of the corollary is similar to a proof of Proposition 3.

Corollary 2. *Let S be an automorphism such that $V_S\boldsymbol{O}(T)=\boldsymbol{O}(T)$ and let W be an automorphism which has each function of $\boldsymbol{O}(T)$ as a proper function. Then S is a Kolmogorov automorphism if and only if $Q=WS$ is a Kolmogorov automorphism.*

Proposition 4. *Let S be an automorphism such that $V_S\boldsymbol{O}(T) = \boldsymbol{O}(T)$ and let W be an automorphism which has each function of $\boldsymbol{O}(T)$ as a proper function. If a totally ergodic automorphism $Q=WS$ has zero entropy, then Q has quasi-discrete spectrum.*

Proof. *Case* (I). $\boldsymbol{O}(T)$ is finitely generated. $B(S)^n\boldsymbol{O}(T)$, $n=1, 2, \cdots$ are subgroups of $\boldsymbol{O}(T)$ and $V_S B^n \boldsymbol{O}(T) = B(S)^n\boldsymbol{O}(T)$. If $B(S)\boldsymbol{O}(T) = \{1\}$, then Q has discrete spectrum, i.e. $\boldsymbol{O}_S(T)_1=\boldsymbol{O}(T)$. If $B(S)\boldsymbol{O}(T) \neq \{1\}$, then there exists a non-trivial σ-algebra $\mathcal{C}(m)$ such that $L^2(\mathcal{C}(m)) = \overline{\text{span}\, B(S)\boldsymbol{O}(T)}$ and $\mathcal{C}(m)$ is invariant under the metric automorphism of S. Suppose now that $\{g \in B(S)\boldsymbol{O}(T) : B(S)g=1 \text{ a.e.}\}=\{1\}$. Then the metric automorphism of S is ergodic on $\mathcal{C}(m)$. This is a contradiction by Proposition 3. Thus we have $\boldsymbol{O}_S(T)_1 \subsetneqq \boldsymbol{O}_S(T)_2$. Next if $B(S)^2\boldsymbol{O}(T)=\{1\}$, then $\boldsymbol{O}_S(T)_2=\boldsymbol{O}(T)$. *If* $B(S)^2\boldsymbol{O}(T)\neq\{1\}$, then we see that $\boldsymbol{O}_S(T)_2 \subsetneqq \boldsymbol{O}_S(T)_3$. It follows from induction to be either $\boldsymbol{O}_S(T)_n = \boldsymbol{O}(T)$ for some integer $n>0$ or $\boldsymbol{O}_S(T)_1 \subsetneqq \boldsymbol{O}_S(T)_2 \subsetneqq \cdots \subsetneqq \boldsymbol{O}_S(T)_n \subsetneqq \cdots$. But we see that there exists an integer n such that $\boldsymbol{O}_S(T)_n=\boldsymbol{O}(T)$ since $O(T)$ is finitely generated. Therefore, by Proposition 2 Q has quasi-discrete spectrum.

Case (II). $\boldsymbol{O}(T)$ is countable, $\boldsymbol{O}(T)=\{g_1, g_2, \cdots, g_n, \cdots\}$. Let $g_k \in \boldsymbol{O}(T)$ has an infinite orbit $\mathrm{O}(g_k)$ under V_S and let $Y(g_k)$ be a subgroup generated by $\mathrm{O}(g_k)$, then $Y(g_k)$ is finitely generated. Because it turns out that $\boldsymbol{O}_1^k \neq \{1\}$ for $j=1$ where $\boldsymbol{O}_j^k=\{g\in Y(g_k) : B(S)^j g=1 \text{ a.e.}\}$, $j=1, 2, \cdots$. Thus for $g \in \boldsymbol{O}_j^k$ with $g\neq 1$ a.e., $V_S g=g$ a.e. and $g=V_S^{n_1}g_k^{\pm1}V_S^{n_2}g_k^{\pm1}\cdots V_S^{n_l}g_k^{\pm1}$a.e. Suppose now that $n_1<n_2<\cdots<n_l$. Then it follows that $Y(g_k)$ is a group generated by $\{g_k, V_S g_k, \cdots, V_S^{n_l}g_k\}$. Let $g_i \in \boldsymbol{O}(T)$ have a finite orbit such that $V_S^n g_i = g_i$ a.e. and let $Y(g_i)$ be a group generated by $\{g_i, V_S g_i, \cdots, V_S^n g_i\}$. Thus we obtain $\boldsymbol{O}(T) = \bigcup_{k=1}^{\infty} Y(g_k)$. By Case (I) there exists an integer j_k such that $Y(g_k)=\boldsymbol{O}_{jk}^k$

$=\bigcup_{j=1}^{\infty}\boldsymbol{O}_j^k$. From $\boldsymbol{O}_S(T)_j=\bigcup_{k=1}^{\infty}\boldsymbol{O}_j^k$, it follows that $\boldsymbol{O}(T)=\bigcup_{k=1}^{\infty}Y(g_k)=\bigcup_{k=1}^{\infty}\bigcup_{j=1}^{\infty}\boldsymbol{O}_j^k$ $=\bigcup_{j=1}^{\infty}\boldsymbol{O}_S(T)_j$. Therefore, by Proposition 2 we have shown that Q has quasi-discrete spectrum.

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