31. *Some Characterizations of Strongly Paracompact Spaces*

By Yoshikazu YASUI
Department of Mathematics, Osaka Kyoiku University

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As is well known,

Theorem 1 (E. Michael [1]). *In a regular T_1-space X, the following properties are equivalent*:

(1) *Every open covering of X has a locally finite open covering as a refinement* (*i.e. X is paracompact*).

(2) *Every open covering of X has a locally finite closed covering as a refinement.*

In this paper, we will characterize the strongly paracompact spaces under the same fashion.

Let us recall the definitions of terms which are used in the statement of this paper. Let X be a topological space and $\mathfrak{A}$ be a collection of subsets of X. The collection $\mathfrak{A}$ is said to be *point finite* (resp. *point countable*) if every point of X is contained in at most finitely (resp. at most countably) many elements of $\mathfrak{A}$. $\mathfrak{A}$ is said to be *locally finite* (resp. *locally countable*) if every point x of X has the neighborhood which intersects only finitely (resp. only countably) many elements of $\mathfrak{A}$. $\mathfrak{A}$ is said to be *star finite* (resp. *star countable*) if every element of $\mathfrak{A}$ intersects only finitely (resp. only countably) many elements of $\mathfrak{A}$. X is said to be *paracompact* (resp. *strongly paracompact*) if every open covering of X has a locally finite (resp. star finite) open covering of X as a refinement.

Finally to state our results we need a next notion. Let $\{U_x \mid x \in X\}$ be a collection of subsets of X with the index set X, then its collection is *symmetric* if "$y \in U_x$" is equivalent to "$x \in U_y$".

We assume that all spaces in this paper are Hausdorff and, for any symmetric collection $\{U_x \mid x \in X\}$, U_x contains the point x for any point $x \in X$.

Theorem 2 (Yu. M. Smirnov [3]). *In a regular space X, the following properties are equivalent*:

(1) *Every open covering of X has a star finite open covering as a refinement* (*i.e. X is strongly paracompact*).

(2) *Every open covering of X has a star countable open covering as a refinement.*

By use of the above theorem, we shall prove the following theorem.

Theorem 3. *In a regular space X, the following properties are equivalent:*

(1) *Every open covering of X has a star finite open covering as a refinement (i.e. X is strongly paracompact).*

(2) *Every open covering of X has a symmetric star finite open collection as a refinement.*

(3) *Every open covering of X has a symmetric locally finite open collection as a refinement.*

(4) *Every open covering of X has a symmetric point finite open collection as a refinement.*

(5) *Every open covering of X has a symmetric star countable open collection as a refinement.*

(6) *Every open covering of X has a symmetric locally countable open collection as a refinement.*

(7) *Every open covering of X has a symmetric point countable open collection as a refinement.*

(8) *Every open covering of X has a symmetric locally finite closed collection as a refinement.*

Proof. (1) *implies* (2). Let $\mathfrak{A}_1$ be an any open covering of X. Clearly the strong paracompactness implies the paracompactness, and therefore implies the full normality[1] by A. H. Stone [4] or J. Nagata [2]. Hence $\mathfrak{A}_1$ has an open covering $\mathfrak{A}_2$ of X as a $\varDelta$-refinement[2] and furthermore $\mathfrak{A}_2$ has a star finite open covering $\mathfrak{A}_3$ of X as a refinement by the strong paracompactness of X.

Let $\quad U_x = \mathrm{St}(x, \mathfrak{A}_3)$[3] $\quad$ for each $x \in X$

and $\quad \mathfrak{U} = \{U_x \mid x \in X\}$,

then $\mathfrak{U}$ will be a symmetric star finite open collection which is a refinement of the given covering $\mathfrak{A}_1$.

At the start, it is evident that $\mathfrak{U}$ is a refinement of $\mathfrak{A}_1$, as $\mathfrak{A}_3$ is a $\varDelta$-refinement of $\mathfrak{A}_1$, and moreover $\mathfrak{U}$ is a symmetric open collection. If we show only the star finiteness of $\mathfrak{U}$, we get the property (2).

For this purpose, in general let $\mathfrak{G}$ be any star finite collection of subsets of X and $\mathfrak{G}^* = \{\bigcup_{i=1}^n G_{\alpha_i} \mid G_{\alpha_i} \in \mathfrak{G}$ for $i = 1, 2, \cdots, n$, $\bigcap_{i=1}^n G_{\alpha_i}\} \neq \emptyset\}$.
At this time, it is sufficient that we prove the star finiteness of $\mathfrak{G}^*$.
Let A be any element of $\mathfrak{G}^*$, i.e.

$$A = \bigcup_{i=1}^n G_{\alpha_i} \quad \text{where } G_{\alpha_i} \in G \quad \text{for } i = 1, 2, \cdots, n; \text{ and } \bigcap_{i=1}^n G_{\alpha_i} \neq \emptyset.$$

1) Topological space X is said to be *fully normal* if every open covering of X has an open covering of X as a $\varDelta$-refinement.[2]

2) Let $\mathfrak{A}$, $\mathfrak{B}$ be coverings of topological space X, then $\mathfrak{A}$ is a *$\varDelta$-refinement* of $\mathfrak{B}$ if $\{\cup\{A \mid A \in \mathfrak{A}, x \in A\} \mid x \in X\}$ is a refinement of $\mathfrak{B}$.

3) $\mathrm{St}(x, \mathfrak{A}_3)$ will denote the union of all the elements of $\mathfrak{A}_3$ which contains x.

If we put $\mathfrak{G}_0=\{G\in\mathfrak{G}\mid G\cap G'\neq\emptyset,\ G'\cap A\neq\emptyset \text{ for some } G'\in\mathfrak{G}\}$,
then by the star finiteness of $\mathfrak{G}$ and the definition of $A, \mathfrak{G}_0$ is finite, and force

$$\mathfrak{G}_0^*=\{\bigcup_{i=1}^{m}\mathfrak{G}_{\beta_i}\mid G_{\beta_i}\in\mathfrak{G}_0\ (i=1,2,\cdots,m),\ \bigcap_{i=1}^{m}G_{\beta_i}\neq\emptyset\} \text{ is finite.}$$

On the other hand,

$$\{B\in\mathfrak{G}^*\mid A\cap B\neq\emptyset\}\subset\mathfrak{G}_0^*$$

and hence we get the star finiteness of $\mathfrak{G}$.

(2) *implies* (3). (3) *implies* (4): It is trivial.

(4) *implies* (8): In this place, we shall show that the point finiteness of any symmetric collection implies the star finiteness of it. Let $\mathfrak{U}=\{U_x\mid x\in X\}$ be any symmetric point finite collection and x_0 be any point of X.
If we put $\mathfrak{U}_x=\{U_y\mid x\in U_y,\ U_y\in\mathfrak{U}\}$ for each $x\in X$,
then $\mathfrak{U}_x$ is finite because $\mathfrak{U}$ is a point finite collection. Therefore we can describe as follows: $\mathfrak{U}_x=\{U_{y_i^{(x)}}\mid i=1,\cdots,n_x\}$ for each $x\in X$.
Let $U_{x_0}\cap U_x\neq 0$ for $U_x\in\mathfrak{U}$, and hence $U_{x_0}\cap U_x\ni y$ for some $y\in X$. Then by the symmetry of $\mathfrak{U}$, we have $x_0\in U_y$, i.e. $U_y=U_{y_{i(x_0)}}$ for some $i\in\{1,\cdots,n_{x_0}\}$.

On the other hand, $x\in U_y=U_{y_{i(x_0)}}$ and hence $y_{i(x_0)}\in U_x$, therefore

$$\{U_x\mid U_x\in\mathfrak{U},\ U_{x_0}\cap U_x\neq\emptyset\}\subset\cup\{\mathfrak{U}_{y_{i(x_0)}}\mid i=1,2,\cdots,n_{x_0}\}$$

where, $\cup\{\mathfrak{U}_{y_{i(x_0)}}\mid i=1,\cdots,n_{x_0}\}$ is finite. Consequently as $\mathfrak{U}=\{U_x\mid x\in X\}$ is a star finite collection, we get the strong paracompactness and therefore the paracompactness and hence the full normality of X under the condition (4).

In order to prove the property (8), let $\mathfrak{A}$ be any open covering of X. From the above discussion, $\mathfrak{A}$ has a symmetric star finite open collection $\mathfrak{U}$ such that $\bar{\mathfrak{U}}$[4] is a Δ-refinement of $\mathfrak{A}$. Let

$$V_x=\mathrm{St}(x,\bar{\mathfrak{U}}) \text{ for each } x\in X \text{ and } \mathfrak{V}=\{V_x\mid x\in X\},$$

then $\mathfrak{V}$ will be symmetric locally finite closed collection. In fact, being the symmetric closed collection is easily seen, and hence we shall show only the local finiteness of $\mathfrak{V}$. At first, let $\bar{U}_x$ be any fixed element of $\bar{\mathfrak{U}}$.
If $\bar{U}_x\cap\bar{U}_y\neq\emptyset$ for $U_y\in\mathfrak{U}$, then there exists $U_z\in\mathfrak{U}$ such that $U_x\cap U_z\neq\emptyset$ and $U_z\cap U_y\neq\emptyset$ because each $U_x\in\mathfrak{U}$ is an open neighborhood of x in X. In conclusion, $\{\bar{U}_y\mid U_y\in\mathfrak{U},\ \bar{U}_x\cap\bar{U}_y\neq\emptyset\}$ is finite for any $\bar{U}_x\in\mathfrak{U}$, from the star finiteness of $\mathfrak{U}$, that is, $\bar{\mathfrak{U}}$ is star finite. Furthermore, as $\bar{\mathfrak{U}}$ is point finite, it is evident for $\mathfrak{V}$ to be star finite.

On the other hand, $V_x\supset\mathrm{St}(x,\mathfrak{U})$ for each $x\in X$ and hence we get the local finiteness of $\mathfrak{V}$.

4) In general, for the collection $\mathfrak{G}$ of subsets of X, $\bar{\mathfrak{G}}$ will denote the collection $\{\bar{G}\mid G\in\mathfrak{G}\}$.

(8) *implies* (1). From Theorem 1 and the property (8), any open covering $\mathfrak{A}$ of X has a symmetric point finite closed covering $\mathfrak{U}$ of X as a Δ-refinement. In a same way as (1) implying (2), $\{\mathrm{St}(x, \mathfrak{U}) \mid x\}$ is a star finite closed covering of X. If we let $V_x = \mathrm{Int}(\mathrm{St}(x, \mathfrak{U}))$,[5] it is evident that V_x contains x by the local finiteness of closed covering $\mathfrak{U}$, and hence $\{V_x \mid x \in X\}$ is a star finite open covering and is a refinement of $\mathfrak{A}$.

(1) *implies* (2). (2) *implies* (5). (5) *implies* (6). (6) *implies* (7): It is trivial.

(7) *implies* (1): In the proof of property (4) implying property (5), we have shown the symmetric point finite collection was the star finite collection. As the same discussions, it is clear that the symmetric point countable collection is star countable. Therefore property (7) implies property (1) by Yu. M. Smirnov [3].

Remark. In property (8) in Theorem 3, we can not give the point finiteness instead of the local finiteness. Actually any regular T_1, and not strongly paracompact space has the property such that any open covering of it has the symmetric point finite open collection as a refinement. The existence of a regular T_1, not strongly paracompact space is well known (see Yu. M. Smirnov [3], or Y. Yasui [5]).

References

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5) $\mathrm{Int}(\mathrm{St}(x, \mathfrak{U}))$ will denote the interior of $\mathrm{St}(x, \mathfrak{U})$ in X.