88. *The Existence and Uniqueness of the Solution of the Equations Describing Compressible Viscous Fluid Flow*

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Since J. Leray's discussion on the nonstationary movement of incompressible viscous fluid, there appeared a number of papers on it such as Kiselev-Ladyzhenskaya's ([1]), but very few reports, if any, on that of compressible viscous fluid have been made, presumably, because of the complexities that the system of equations describing it contains. In view of these circumstances, we try to find a way of solving this problem firstly from a classical point of view.

1. Introduction. When μ(viscosity), k(heat conductivity), and c_v(specific heat at constant volume) are constants (which does not injure the mathematical generality), the movement of isotropic Newtonian fluid is described as follows: (ρ : density, v : velocity, f : outer force, p : pressure, θ : absolute temperature, and $F(\nabla v)$: dissipation function (≥ 0)),

$$(1.1)\quad \begin{cases} \dfrac{\partial \rho}{\partial t}+\operatorname{div}\rho v=0, \\ \rho\left(\dfrac{\partial v}{\partial t}+(v\cdot\nabla)v\right)=\rho f-\nabla\rho\dfrac{\partial p}{\partial \rho}-\nabla\theta\dfrac{\partial p}{\partial\theta}+\left(\mu\Delta+\dfrac{\mu}{3}\cdot\nabla\operatorname{div}\right)v, \\ c_v\rho\dfrac{\partial\theta}{\partial t}=k\Delta\theta+F-\theta\dfrac{\partial p}{\partial\theta}\operatorname{div}\rho v-c_v\rho v\cdot\nabla\theta, \end{cases}$$

$$(1.1)'\quad p=\theta\sum_{n=1}^{\infty}\hat{a}_n\rho^n,\ (0<\rho<\rho^*=\text{radius of convergence of } p;\ p \text{ is assumed to be virially expanded}).$$

We shall consider a Cauchy problem of (1.1) in which the initial condition is given by

$$(1.1)''\quad \rho(x,0)=\rho_0(x)(>0),\ v(x,0)=v_0(x),\ \theta(x,0)=\theta_0(x)(\geq 0).$$

In the first place, we make the following linear problem correspond with the 2nd expression of (1.1).

$$(1.2)\quad \begin{cases} \dfrac{\partial v}{\partial t}=\sigma(x,t)\left(\Delta+\dfrac{1}{3}\nabla\operatorname{div}\right)v+f\equiv\sigma(x,t)P_0(D_x)v+f, \\ v(x,0)=v_0(x)(\in H^{2+\alpha}(\overline{R^3})),\ ((x,t)\in R^3_T), \end{cases}$$

where σ, $f\in H^\alpha_T$, $0<\sigma_0\leq\sigma(x,t)\leq\sigma_1<+\infty$, and H^α_T is the space of functions $g(x,t)$ defined on $\overline{R^3_T}(R^3_T\equiv R^3\times(0,T))$ such that

$$\begin{cases} |g|_T^{(0)} \equiv \sup_{R_T^3} |g| < +\infty, \ |g|_T^{(\alpha)} \equiv |g|_{x,T}^{(\alpha)} + |g|_{t,T}^{(\alpha/2)} \\ \qquad \equiv \sup \dfrac{|g(x,t)-g(x',t)|}{|x-x'|^\alpha} + \sup \dfrac{|g(x,t)-g(x,t')|}{|t-t'|^{\alpha/2}} < +\infty. \end{cases}$$

The system (1.2) is uniformly parabolic in Petrowsky's sense, i.e.,

$$\exists a_0 > 0 \text{ such that } \max_j \sup_{|\xi|=1} \operatorname{Re} \beta_j(\xi; x, t) < -a_0(\forall (x,t) \in R_T^3),$$

where β_j's are the roots of $\det(\sigma P_0(i\xi) - I\beta)$ and $\xi \in R^3$.

2.1. The fundamental solution and its estimates. In relation with (1.2), we consider the following system of ordinary differential equations,

$$\text{(2.1)} \qquad \frac{dV}{dt} = \sigma(y,s)P_0(i\zeta)V(\zeta,t;y,s),\ V|_{t=s} = I(\text{unit matrix}),\ (\zeta \in C^3)$$

and define $Z = (Z^{ij})$, using V as follows: $(i, j = 1, 2, 3)$

$$\text{(2.2)} \qquad Z^{ij}(x-z,t;y,s) \equiv (2\pi)^{-3} \int_{R^3} \exp\,[i\xi_0(x-z)] V^{ij}(\xi_0,t;y,s)\,d\xi_0.$$

As for Z^{ij}, we have, e.g., for $m(|m| \geq 0)$, and $t > s$: (m, index vector),

$$\text{(2.3)} \qquad \begin{aligned} &|D_x^m Z^{ij}(x-z,t;y,s)| \\ &\leq C_1^{(|m|)}(t-s)^{-(|m|+3)/2} \exp\left[-(24a_1\sigma_1)^{-1}\frac{|x-z|^2}{t-s}\right], \end{aligned}$$

(a_1 depends on P_0).

There exists a unique bounded solution of (1.2). Thus, the so-called fundamental solution Γ for it is unique, and has an expression

$$\text{(2.4)} \qquad \begin{aligned} \Gamma(x,t;z,s) &= Z(x-z,t;z,s) \\ &+ \int_0^t dt_0 \int_{R^3} Z(x-z,t;y,s_0) J(y,s_0;z,s)\,dy, \end{aligned}$$

where J satisfies a Volterra type integral equation. For J, we have:

$$\text{(2.5)} \qquad |J(x,t;z,s)| \leq C_2(t-s)^{-5-\alpha/2} \exp\left[-\bar{C}_2 \frac{|x-z|}{t-s}\right].$$

By (2.3), (2.5), etc., it is shown that J is Hölder-continuous with the exponents α and $\alpha/2$ in x and t, respectively.

2.2. Estimates of the bounded solution of a linear problem. As $v_0(x) \in H^{2+\alpha}(\overline{R^3})$ in (1.2),

$$\text{(2.6)} \qquad v(x,t) = v_0(x) + \int_0^t ds \int_{R^3} \Gamma(x,t;z,s)[\sigma(z;s)P_0(D_z)v_0(z) + f(z,s)]\,dz.$$

Finally, we have, e.g., for $|m| = 1, 2$:

$$\text{(2.7)} \qquad \begin{aligned} &|D_x^m v(x,t)| \leq |D_x^m v_0(x)| + C_5^{(|m|)} t^{(2-|m|+\alpha)/2} \| f + \sigma P_0 v_0 \|_T^{(\alpha)}, \\ &(|m| = 1, 2;\ \|\cdot\|_T^{(\alpha)} \equiv |\cdot|_{R_T^3}^{(0)} + |\cdot|_{R_T^3}^{(\alpha)}), \end{aligned}$$

$$\text{(2.7)}' \qquad |D_x^m v(x,t) - D_x^m v(x,t')| \leq C_6^{(|m|)} |t-t'|^{(2-|m|+\alpha)/2} \| f + \sigma P_0 v_0 \|_T^{(\alpha)}.$$

$C_5^{(|m|)}$, etc., are positive functions continuous in $\sigma_0, \sigma_1, |\sigma|_T^{(\alpha)}$, and T, and monotonically increasing in each parameter.

The equation connected with the 3rd expression of (1.1),

$$\text{(2.8)} \qquad \frac{\partial \theta}{\partial t} = \tilde{\sigma}(x,t)\Delta\theta + g,\ (\tilde{\sigma}, g \in H_T^\alpha,\ 0 < \sigma_0 \leq \tilde{\sigma} \leq \tilde{\sigma}_1 < +\infty),$$

is, of course, uniformly parabolic in Petrowsky's sense. So, we can treat (2.8) in parallel with (1.2). The corresponding constants, etc., will be denoted, e.g., by $'C_1$, etc.

3.1. The existence of a bounded solution of (1.1)-(1.1)″. We define:

$$(3.1)\quad \begin{cases} \langle u\rangle_T \equiv \sum_{|m|=0}^{2} |D_x^m u|_{R_T^3}^{(0)} + \sum_{|m|=0}^{1} |D_x^m u|_{t,R_T^3}^{(\alpha/2)}, \qquad \langle u\rangle_T' \equiv \sum_{|m|=2} |D_x^m u|_{R_T^3}^{(\alpha)}, \\ (\theta)_T \equiv \sum_{|m|=0}^{1} (|D_x^m\theta|_{R_T^3}^{(0)} + |D_x^m\theta|_{t,R_T^3}^{(\alpha/2)}) + \sum_{|m|=1} |D_x^m\theta|_{x,R_T^3}^{(\alpha)}, \\ \langle\!\langle u\rangle\!\rangle_T \equiv \langle u\rangle_T + \langle u\rangle_T', \hat{H}_T^{2+\alpha} \equiv \{u : \langle\!\langle u\rangle\!\rangle_T < +\infty\}, H_T^{2+\alpha} \\ \qquad \equiv \{u : \langle\!\langle u\rangle\!\rangle_T + \|D_t u\|_T^{(\alpha)} < +\infty\}, \hat{H}_T^{1+\alpha} \equiv \{\theta : (\theta)_T < +\infty\}. \end{cases}$$

If $v \in \hat{H}_T^{2+\alpha}$ in the first expression of (1.1), then ρ is expressed by

$$(3.2)\qquad \rho(x,t) = \rho_0(x_0(x,t)) \exp\left[-\int_0^t \operatorname{div} v(\bar{x}(s\,;x,t),s)ds\right],$$

where $\bar{x}(s\,;x,t)$ is the characteristic curve passing (x,t) and $x_0(x,t) = \bar{x}(0\,;x,t)$. It is easily shown that the correspondence $\{x_0 = x_0(x,t), t_0 = t\}$ is $1-1$, and that $D_{x_0}^m x, D_x^m x_0$, etc. $(|m| \leq 2)$ exist, being bounded and continuous. For $\theta < 0$, we define p by the righthand side of (1.1)′. We assume in (1.1)″ that

$$(3.3)\qquad \rho_0 \in H^{1+\alpha}(\overline{R^3}),\ (0 < \bar{\rho}_0 \leq \rho_0 \leq \bar{\bar{\rho}}_0 < \rho^*),\ \theta_0 \text{ and } v_0 \in H^{2+\alpha}(\overline{R^3}).$$

We take ${}^\forall M_1 > \|v_0\|_{R^3}^{(2)}$, and choose T_1 such that $0 < T_1 < M_1^{-1} \log(\rho^*/\bar{\bar{\rho}}_0)$. Now, we define a mapping G_{T_1} from

$$(3.4)\qquad \begin{aligned} S_{T_1}^0 \equiv \{(v,\theta) : (v,\theta) \in \hat{H}_{T_1}^{2+\alpha} \times \hat{H}_{T_1}^{1+\alpha}, \langle v\rangle_{T_1} \\ \leq M_1, v(x,0) = v_0, \theta(x,0) = \theta_0\} \end{aligned}$$

into $\hat{H}_{T_1}^{2+\alpha} \times \hat{H}_{T_1}^{1+\alpha}$ in the following way:

$$(3.5)\quad \begin{cases} \hat{v}(x,t) = v_0(x) + \int_0^t ds \int_{R^3} \Gamma(x,t\,;z,s\,;\rho_v) \left\{N_1 + \dfrac{\mu}{\rho_v} P_0 v_0\right\}(z,s)dz, \\ \hat{\theta}(x,t) = \theta_0(x) + \int_0^t ds \int_{R^3} {}'\Gamma(x,t\,;z,s\,;\rho_v) \left\{N_2 + \dfrac{k}{c_v\rho_v} \Delta\theta_0\right\}(z,s)dz, \end{cases}$$

$$\left(N_1 = f - \frac{\nabla\rho}{\rho}\frac{\partial p}{\partial\rho} - \frac{\nabla\theta}{\rho}\frac{\partial p}{\partial\theta} - (v\cdot\nabla)v, N_2 = \frac{F}{c_v\rho} - \frac{p}{c_v\rho}\operatorname{div} v - v\cdot\nabla\theta\right).$$

G_{T_1} is well-defined, because the following inequalities hold:

$$(3.6)\quad \begin{cases} \left\|N_1 + \dfrac{\mu}{\rho_v} P_0 v_0\right\|_{R_{T_0}^3}^{(\alpha)} \leq B_1(\langle v\rangle_{T_0}, T_0) + B_2(\langle v\rangle_{T_0}, (\theta)_{T_0}, T_0) \\ \qquad\qquad \times \langle v\rangle_{T_0}', (B_2 \searrow 0 (T_0 \searrow 0)), \\ \left\|N_2 + \dfrac{k}{c_v\rho_v}\Delta\theta_0\right\|_{R_{T_0}^3}^{(\alpha)} \leq B_3(\langle v\rangle_{T_0}, (\theta)_{T_0}, T_0), ({}^\forall T_0 \in [0, T_1]). \end{cases}$$

By (2.7), etc., we obtain from (3.5):

$$(3.7)\quad \begin{cases} \langle\hat{v}\rangle_{T_0} \leq \|v_0\|_{R^3}^{(2)} + A(\langle v\rangle_{T_0}, T_0)[B_1(\langle v\rangle_{T_0}, (\theta)_{T_0}, T_0) \\ \qquad\qquad + B_2(\langle v\rangle_{T_0}, (\theta)_{T_0}, T_0)(v)_{T_0}'], \\ (\hat{\theta})_{T_0} \leq \|\theta_0\|_{R^3}^{(1+\alpha)} + {}'A(\langle v\rangle_{T_0}, T_0) \times B_3(\langle v\rangle_{T_0}, (\theta)_{T_0}, T_0), \\ \langle\hat{v}\rangle_{T_0}' \leq |\nabla\nabla v|_{R^3}^{(\alpha)} + \hat{A}(\langle v\rangle_{T_0}, T_0) \times [B_1(\langle v\rangle_{T_0}, (\theta)_{T_0}, T_0) \\ \qquad\qquad + B_2(\langle v\rangle_{T_0}, (\theta)_{T_0}, T_0)\langle v\rangle_{T_0}'], \end{cases}$$

where A, B_2, and $'A\searrow 0$ $(T_0\searrow 0)$. Taking $M_2>\|\theta_0\|_{R^3}^{(1+\alpha)}$, we can choose $T'\in(0,T_1]$ and $M_3(>0)$ such that $\langle v,\hat{v}\rangle_{T'}\le M_1$, $\langle\theta,\hat{\theta}\rangle_{T'}\le M_2$, $\langle v,v\rangle'_{T'}\le M_3(M_1,M_2)$. Thus, if we denote by $S_{T'}$ the set of $(v,\theta)\in S^0_{T'}$ such that $(v,0)$ satisfies the above-cited condition, then

(3.8) $$G_{T'}S_{T'}\subset S_{T'}\cap(H^{2+\alpha}_{T'}\times H^{2+\alpha}_{T'})\subset S_{T'}\subset\dot{H}^{2+\alpha\beta}_{T'}\times\dot{H}^{1+\alpha\beta}_{T'},\ (\beta\in(0,1)),$$

where $\dot{H}^{2+\alpha\beta}_{T'}\times\dot{H}^{1+\alpha\beta}_{T'}$ is a Fréchet space defined by a countable system of seminorms

(3.9) $$[(v,\theta)]_{N,T'}\equiv\langle\!\langle v\rangle\!\rangle_{N,T'}+\langle\theta\rangle_{N,T'},\ (N=1,2,\cdots),$$

where the suffix "N,T'" indicates that "sup" is considered in $R^3_{N,T'}\equiv\{(x,t):|x|\le N+M_1(T'-t),0\le t\le T'\}$. After some calculations, it is shown that $S_{T'}$ is a compact convex subset in $\dot{H}^{2+\alpha\beta}_{T'}\times\dot{H}^{1+\alpha\beta}_{T'}$, and that $G_{T'}$ is a continuous operator from $S_{T'}$ as a subset of $\dot{H}^{2+\alpha\beta}_{T'}\times\dot{H}^{1+\alpha\beta}_{T'}$ into itself. Therefore, by Tikhonov's theorem on the existence of a fixed point in a locally convex linear topological space, we have:

Theorem. *For some $T'\in(0,T]$, $\exists(v,\theta,\rho)\in H^{2+\alpha}_{T'}\times H^{2+\alpha}_{T'}\times\tilde{B}^{1+\alpha}_{T'}$ such that (v,θ,ρ) satisfies (1.1)-(1.1)″. Moreover, $\theta(x,t)\ge 0$.*

$(\tilde{B}^{1+\alpha}_{T'}=\{\rho:|\rho|^{(0)}_{T'}+|v\rho|^{(\alpha)}_{T'}+|D_t\rho|^{(\alpha)}_{T'}\}$; T', *used for* $R^3_{T'}$).

3.2. The problem of uniqueness. Assuming that there are two solutions, we estimate their difference on the basis of the expression that it satisfies. More precision is needed than in the case of Theorem 1.

Theorem 2. *If, for $\rho_0\in H^{2+L}(\bar{R}^3)$, there exists a solution (v,θ,ρ) of (1.1)-(1.1)″ $\in(H^{2+\alpha}_T\cap B^{3+L}_{x,T})\times H^{2+\alpha}_T\times\tilde{B}^{1+\alpha}_T$, then the solution is unique there. ('L' denotes that Lipschitz's condition is satisfied).*

If the given functions are smoother, then, taking account of Theorem 2, we have:

Theorem 3. *If*

$$\rho_0\in H^{2+L}(\bar{R}^3)(0<\bar{\rho}_0\le\rho_0),\ v_0\in H^{4+\alpha}(\bar{R}^3),\ \theta_0(x)\in H^{3+\alpha}(\bar{R}^3),$$

and $f\in H^{2+\alpha}_T$, then, for some $T'\in(0,T]$, $\exists_1(v,\theta,\rho)\in H^{4+\alpha}_{T'}\times H^{3+\alpha}_{T'}\times\tilde{B}^{1+\alpha}_{T'}$ such that (v,θ,ρ) satisfies (1.1)-(1.1)″.

$$\left(H^{n+\alpha}_{T'}\equiv\left\{g:\sum_{2r+|m|=0}^{n}|D^r_tD^m_xg|^{(0)}_{R^3_{T'}}+\sum_{2r+|m|=n-1}^{n}|D^r_tD^m_xg|^{(\alpha/2)}_{t,R^3_{T'}}+\sum_{2r+|m|=n}|D^r_tD^m_xg|^{(\alpha)}_{x,R^3_{T'}}<+\infty\right\}(n=3,4)\right).$$

Referenecs

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