18. Theorems on the Finite-dimensionality of Cohomology Groups. I

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The purpose of this note is to present some theorems on the finite-dimensionality of cohomology groups with solution sheaves of (systems of) linear differential equations as their coefficients. Their proofs and detailed arguments will be published somewhere else and here we only remark that they are deduced in a unified manner by the "comparison principle" of the cohomology groups combined with the regularity property of the solutions of linear differential equations under considerations (Kawai [3]). Note that the regularity property is a local property of linear differential equations. The application of the results of this note to extend the fixed point formulas of Lefschetz-Atiyah and Bott (Atiyah and Bott [1]) and the extension of the results by the aid of pseudo-differential equations on the boundary suitably induced by the (systems of) linear differential equations under consideration will be discussed in our forthcoming notes. We also note that in a very recent paper [2] Guillemin has announced results close to ours by the aid of sub-elliptic estimates for complexes, while our proofs mainly rely on the classical existence theorem of Cauchy and Kowalevsky for systems of linear differential equations. Throughout this note we consider the problems exclusively in real analytic category, that is, "manifold" means "real analytic manifold", the linear differential equation under consideration has real analytic coefficients and so forth. In the sequel we denote by $\mathcal{A}$ the sheaf of germs of real analytic functions, by $\mathcal{B}$ that of hyperfunctions and by $\mathcal{D}$ that of linear differential operators. In the sequel we regard $\mathcal{A}$ and $\mathcal{B}$ as sheaves of left $\mathcal{D}$-modules. As is usual $\mathcal{B}/\mathcal{A}$ denotes the quotient sheaf of $\mathcal{B}$ by $\mathcal{A}$. We also abbreviate $\mathcal{A}^r, \mathcal{B}^r$ and $\mathcal{D}^r$ to $\mathcal{A}, \mathcal{B}$ and $\mathcal{D}$ respectively for short, where $\mathcal{A}^r, \mathcal{B}^r$ and $\mathcal{D}^r$ denote the sheaf of germs of r-tuples of real analytic functions, that of hyperfunctions and that of linear differential operators respectively.

Theorem 1. *Let $\mathfrak{M}$ be a system of linear differential equations on a compact manifold M (without boundary). Assume that $\mathfrak{M}$ satisfies the following two conditions* (1) *and* $(2)_k$. *(See Sato, Kawai and Kashiwara* [4] *as for condition* $(2)_k$*). Then we have*

$$\dim_C \mathrm{Ext}^j (M, \mathfrak{M}, \mathcal{B}) < \infty \quad \text{for} \quad j=0, \cdots, k,$$

where $\mathrm{Ext}^j(M, \mathfrak{M}, \mathcal{B})$ *denotes*

$$\mathrm{Ker}\,(P_j: \mathcal{B}(M) \to \mathcal{B}(M)) / \mathrm{Im}\,(P_{j-1}: \mathcal{B}(M) \to \mathcal{B}(M)).$$

(1) *The system* $\mathfrak{M}$ *of linear differential equations admits a free resolution of length* $(k+1)$, *i.e., there exist linear differential operators* $\{P_j(x, \partial/\partial x)\}_{j=0}^{k}$ *defined on* M *such that*

$$0 \longleftarrow \mathfrak{M} \longleftarrow \mathcal{D} \xleftarrow{P_0} \mathcal{D} \longleftarrow \cdots \xleftarrow{P_k} \mathcal{D}.$$

$(2)_k$ $$\mathcal{E}xt^j_{\mathcal{D}}(\mathfrak{M}, \mathcal{B}/\mathcal{A}) = 0 \quad \text{for} \quad j = 0, \cdots, k.$$

Remark 1. If the real analytic solution sheaf $\mathcal{S}$ of the system $\mathfrak{M}$, i.e., $\mathcal{H}om_{\mathcal{D}}(\mathfrak{M}, \mathcal{A})$ admits a resolution of length $(k+1)$ by sheaf $\mathcal{A}$, which is not a severe one from the analytical points of view by the classical existence theorem of Cauchy and Kowalevsky, then we have as a trivial corollary of Theorem 1

$$\dim_C H^j(M, \mathcal{S}) < \infty \quad \text{for} \quad j = 0, \cdots, k.$$

Remark 2. Denote by $\mathcal{A}(M)$ and $\mathcal{B}(M)$ the space of all real analytic functions on M and space of all hyperfunctions on M respectively. Then $\mathcal{A}(M)$ and $\mathcal{B}(M)$ can be endowed with the structure of DFS-space and that of FS-space respectively in a natural way since we have assumed that M is compact. Under this topological structure $P_j(x, \partial/\partial x)\mathcal{A}(M)$ and $P_j(x, \partial/\partial x)\mathcal{B}(M)$ are closed in $\mathcal{A}(M)$ and $\mathcal{B}(M)$ respectively for $j = 0, \cdots, k-1$. This fact, which is of course a trivial corollary of Theorem 1, is proved in the course of the proof of Theorem 1.

Theorem 2. *Let* M *be a relatively compact open submanifold (with smooth boundary) of a manifold* N. *Denote by* ∂M *the boundary of* M. *Assume that a system* $\mathfrak{M}$ *of linear differential equations defined on* N *satisfies the following conditions* (3), (4) *and* (5). *Then we have*

$$\dim_C H^j(M, \mathcal{S}) < \infty \quad \text{for any} \quad j \geqq 0,$$

where $\mathcal{S}$ *denotes the real analytic solution sheaf of the system* $\mathfrak{M}$, *i.e.,* $\mathcal{H}om_{\mathcal{D}}(\mathfrak{M}, \mathcal{A})$.

(3) *The solution sheaf* $\mathcal{S}$ *of* $\mathfrak{M}$ *admits a following resolution by sheaf* $\mathcal{A}$:

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{A} \xrightarrow{P_0} \mathcal{A} \xrightarrow{P_1} \cdots \xrightarrow{P_s} \mathcal{A} \longrightarrow 0$$

by the aid of linear differential operators $\{P_j(x, \partial/\partial x)\}$ *defined on* N.

(4) *The system* $\mathfrak{M}$ *of linear differential equations is elliptic in* N.

(5) *The tangential system* $i^*\mathfrak{M}$ *of linear differential equations defined on* ∂M, *which is by definition the system induced from* $\mathfrak{M}$ *by the natural injection* $i: \partial M \to N$, *is elliptic on* ∂M.

Remark. This theorem may be regarded as a partial generalization of a recent result of P. Schapira on the finiteness of the index of linear differential operator $P(z, \partial/\partial z)$ with holomorphic coefficients in a complex domain Ω (to appear in a report of Katata symposium), which was the first motivation of the present writer's investigation of the finite dimensionality of cohomology groups whose coefficients are the solution sheaves of linear differential equations.

Theorem 3. *Let M be a compact submanifold of a manifold N. Assume that a system $\mathfrak{M}$ of linear differential equations defined on N satisfies conditions* (3) *and* (4) *in Theorem* 2 *and condition* (6) *below. Then we have*

$$\dim_C H^j_M(N, \mathcal{S}) < \infty \quad \text{for any} \quad j \geqq 0,$$

where $\mathcal{S}$ denotes the solution sheaf $\mathcal{H}om_{\mathcal{D}}(\mathfrak{M}, \mathcal{A})$.

$$(6)\quad \left\{\begin{array}{l}\textit{The tangential system } j^*\mathfrak{N} \textit{ of linear differential equations de-}\\ \textit{fined on } M\textit{, which is induced by the injection } j: M \to N \textit{ from the}\\ \textit{formal adjoint system } \mathfrak{N} \textit{ of } \mathfrak{M}\textit{, is elliptic on } M.\end{array}\right.$$

Remark 1. This theorem can be extended to the case where M has a smooth boundary $\partial_V M$ in a submanifold V of N if we assume further the ellipticity of the system induced on $\partial_V M$ from the system $\mathfrak{N}$.

Remark 2. Mr. Kashiwara remarked to the present writer that the ellipticity of the system $j^*\mathfrak{N}$ is equivalent to that of $j^*\mathfrak{M}$.

References

[1] Atiyah, M. F., and R. Bott: A Lefschetz fixed point formula for elliptic complexes. I. Ann. of Math., **86**, 374–407 (1967).

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[3] Kawai, T.: Fundamental principle on the finiteness of cohomology groups. RIMS-96. Technical Report of R.I.M.S. Kyoto Univ. (1971) (in Japanese).

[4] Sato, M., T. Kawai, and M. Kashiwara: On pseudo-differential equations in hyperfunction theory (to appear in Proc. A.M.S. symposium on the theory of partial differential equations).