11. *On Some Subgroups of the Group $PSp(2n, q)$*

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Introduction. We say that a subgroup H of a group G is of rank 2 if the number of double cosets $H\backslash G/H$ is equal to 2. Any subgroup of rank 2 of G is the stabilizer of a point of some doubly transitive permutation representation of G, and vice versa.

In [2] the author has determined the rank 2 subgroups of the symplectic group $Sp(2n, 2)$. After that, a remarkable paper of G. M. Seitz [7] has been appeared, which determined all flag transitive subgroups of the finite Chevalley groups (among which all simple groups of Lie type are included) and which asserts together with the result of J. A. Green [5] that there exists an integer N (which depends only on the type of the associated Weyl group) such that if $q \geqslant N$ then rank 2 subgroups of the Chevalley group of the type defined over the finite field $GF(q)$ are among the maximal parabolic subgroups. However, any effective bound for the integer N is not obtained at present.

The purpose of this note is to give an outline of the proof of the following Theorem 1. The proof is done combining the idea used in [2] and the main result given in Seitz [7]. Details will be published elsewhere together with that of [2].

Theorem 1. *The simple group $PSp(2n, q)$, $n \geqslant 10$ and $q > 2$, has no subgroup of rank* 2.

Remark. The assumption that $n \geqslant 10$ is not a serious one but a mere convention not to make the argument so complicated, and it is possible to loosen the restriction a little, say up to $n=7$ (or 6). Nevertheless for small values of n some additional special treatments are needed, and it is not yet done completely at the time of writing this note. However, it will be not so difficult to settle those remaining cases. Our method given here is also applicable for *any* finite Chevalley groups with high rank (as BN-pair)(it is sufficient if we take $n \geqslant 15$, say). These will be treated in a subsequent paper.

§1. The group $Sp(2n, q)$. We may define $G = Sp(2n, q)$, the symplectic group defined over the finite field $GF(q)$, by

$$G = \left\{ X \in GL(n, q)\, ;\ {}^tXJX = J, \ \text{with} \ J = \begin{pmatrix} & -I_n \\ I_n & \end{pmatrix} \right\}.$$

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Here I_n denotes the $n\times n$ identity matrix, and the unwritten places of any matrix always represents 0. The group $Sp(2n, 2^r)(=PSp(2n, 2^r))$ is simple if $n\geqslant 3$, or $n=2$ and $r>1$, while $Sp(2n, p^r)$ for odd prime p contains a normal subgroup of order 2 consisting of I_{2n} and $-I_{2n}$, and the factor group $PSp(2n, p^r)=Sp(2n, p^r)/\{\pm I_{2n}\}$ is simple for $n\geqslant 2$.

In order to prove the non-existence of rank 2 subgroups of $PSp(2n, q)$ we have only to prove the non-existence of rank 2 subgroups of the group $Sp(2n, q)$. Thus in the rest of this note we consider about the group $Sp(2n, q)$.

Let us define some subgroups of the group G as follows:

$$Q=\left\{X\in GL(2n, q)\,;\, X=\begin{pmatrix} I_n & B \\ & I_n\end{pmatrix} \text{ with } {}^tB=B\right\},$$

$$L=\left\{X\in GL(2n, q)\,;\, X=\begin{pmatrix} A & \\ & {}^tA^{-1}\end{pmatrix} \text{ with } A\in GL(n, q)\right\},$$

$$R=\left\{X\in GL(2n, q)\,;\, X=\begin{pmatrix} A & \\ & {}^tA^{-1}\end{pmatrix} \text{ where } A \text{ is any upper triangular } n\times n \text{ matrix}\right\}.$$

Then L and R normalize Q. We set $B=RQ$ (semi-direct product).

We can also regard the group $G=Sp(2n, q)$ as the Chevalley group of type (C_n) defined over the field $GF(q)$. Naturally G has a Tits system (i.e., BN-pair) whose Coxeter diagram (W, R) is as follows:

$$\underset{w_1}{\circ}\!-\!\!-\!\underset{w_2}{\circ}\!-\!\!-\!\underset{w_3}{\circ}\!-\!\!-\cdots-\!\!-\!\underset{w_{n-1}}{\circ}\!=\!=\!\underset{w_n}{\circ}, \qquad R=\{w_1, w_2, \cdots, w_n\}.$$

For any subset $J\subset R$, the groups W_J and G_J are defined by

$W_J=$the group generated by the w_i with $w_i\in J$,

$G_J=\bigcup_{w\in W_J} BwB$, where B denotes the Borel subgroup of the Tits system. Taking a suitable identification of $Sp(2n, q)$ between linear group and Chevalley group, we may take as follows:

the subgroup $RQ=B$ for the Borel subgroup,

the group $L^{(1)}Q$ for the parabolic subgroup $G_{R-\{w_1\}}$, where

$$L^{(1)}=\left\{X\in GL(2n, q)\,;\, X=\begin{pmatrix} A & \\ & {}^tA^{-1}\end{pmatrix} \text{ with } A=\begin{pmatrix} * & * \\ 0 & \\ \vdots & * \\ 0 & \end{pmatrix}\in GL(n, q)\right\},$$

the group $L^{(2)}Q$ for the parabolic subgroup $G_{R-\{w_2\}}$, where

$$L^{(2)}=\left\{X\in GL(2n, q)\,;\, X=\begin{pmatrix} A & \\ & {}^tA^{-1}\end{pmatrix} \text{ with } A=\begin{pmatrix} * & * & * \\ * & * & \\ 0 & 0 & \\ \vdots & \vdots & * \\ 0 & 0 & \end{pmatrix}\in GL(n, q)\right\}.$$

Moreover we have for the element $u_1=I_n+e_{1,n+1}$, $C_G(u_1)=L^{(1)\prime}Q$

$(\subset G_{R-\{w_1\}})$, where $L^{(1)\prime}=\left\{X\in GL(2n,q)\,;\ X=\begin{pmatrix}A & \\ & {}^tA^{-1}\end{pmatrix}\text{ with } A=\begin{pmatrix}1 & * \\ 0 & \\ \vdots & * \\ 0 & \end{pmatrix}\right.$

$\left.\in GL(n,q)\right\}$. If q is a power of 2, for the element $u_2=I_{2n}+e_{1,n+2}+e_{2,n+1}$, $C_G(u_2)=L^{(2)\prime}Q(\subset G_{R-\{w_2\}})$, where $L^{(2)\prime}=\left\{X\in GL(2n,q)\,;\ X=\begin{pmatrix}A & \\ & {}^tA^{-1}\end{pmatrix}\text{ with } A=\begin{pmatrix}C & D\\ & E\end{pmatrix},\ C\in SL(2,q),\ E\in GL(n-2,q)\right\}$.

§2. **Outline of the proof of Theorem 1.** Let H be a subgroup of rank 2 of the group $G=Sp(2n,q)$, and let χ be the irreducible character of G such that $(1_H)^G=1_G+\chi$, where 1_H and 1_G denote the identity characters of H and G respectively and $(1_H)^G$ denotes the induced character of 1_H to G. We fix these notations throughout this note.

Our essential point of the proof is to show that χ appears in some $(1_P)^G$ for some relatively large parabolic subgroup P of G. (In [7] it is asserted that χ appears in $(1_B)^G$.)

In the rest of this note, we always assume that $n\geqslant 10$ and $q>2$ unless the contrary is stated.

Lemma 1. *$|G:H|\leqslant q^{2n}$, consequently $\chi(1)\leqslant q^{2n}-1$.*

Proof. Since $|G:C_G(u_1)|=q^{2n}-1$, we have the assertion by a lemma of Ed. Maillet (cf. [1], Lemma 3).

Lemma 2. *Taking a suitable conjugate H^x of H, $H^x\cap L$ contains a subgroup K of the form*

$$K=\left\{X=\begin{pmatrix}A & \\ & {}^tA\end{pmatrix}\in L\,;\ A=\begin{pmatrix}I_i & & \\ & C & \\ & & I_{2-i}\end{pmatrix}\ \text{with}\ C\in SL(n-2,q)(\text{for some } 0\leq i\leq 2)\right\}.$$

To prove Lemma 2, we need Propositions A and B.

Proposition A. (This is proved by making use of the results in J. A. Green [4]. Here we use the assumption that $n\geqslant 10$.) *The irreducible characters of $GL(n,q)$ whose degrees are $\leqslant q^{2n}$ are multiples of 1 or -1 of the following (generalized) characters:*

1) $I_1^{k_1}[n]$, of degree 1,
2) $I_1^{k_1}[n-1,1]$, of degree $q\dfrac{q^{n-1}-1}{q-1}$,
3) $I_1^{k_1}[n-2,2]$, of degree $q^2\dfrac{(q^n-1)(q^{n-1}-1)}{(q-1)(q^2-1)}$,

4) $I_1^{k_1}[n-2,1,1]$, of degree $q^3\dfrac{(q^{n-1}-1)(q^{n-2}-1)}{(q-1)(q^2-1)}$,

5) $I_1^{k_1}[1]\circ I_2^{k_2}[n-1]$, of degree $\dfrac{q^n-1}{q-1}$,

6) $I_1^{k_1}[1]\circ I_1^{k_2}[n-2,1]$, of degree $q\dfrac{(q^n-1)(q^{n-2}-1)}{(q-1)(q-1)}$,

7) $I_1^{k_1}[2]\circ I_1^{k_2}[n-2]$, of degree $q\dfrac{(q^n-1)(q^{n-1}-1)}{(q-1)(q^2-1)}$,

8) $I_1^{k_1}[1,1]\circ I_1^{k_2}[n-2]$, of degree $q\dfrac{(q^n-1)(q^{n-1}-1)}{(q-1)(q^2-1)}$,

9) $I_2^{k_1}[1]\circ I_1^{k_2}[n-2]$, of degree $\dfrac{(q^n-1)(q^{n-1}-1)}{q^2-1}$,

10) $I_1^{k_1}[1]\circ I_1^{k_2}[1]\circ I_1^{k_3}[n-2]$, of degree $\dfrac{(q^n-1)(q^{n-1}-1)}{(q-1)(q-1)}$.

(For the notation, see [4]. Here the simpleces k_i run suitable range.)

Proposition B. (This is also proved by using the results of Green [4]. Here we also use the assumption that $n\geqslant 10$.) *For every irreducible character φ of $GL(n,q)$ listed in Proposition* A, *we have for the element $y=I_n+e_{1,n}\in GL(n,q)$,*

$$\varphi(1)<\varphi(y)\cdot q^{n-3}.$$

Proof of Lemma 2. From Proposition B we can conclude that $H^x\cap L$ contains two elements of $L\left(\cong GL(n,q)\right.$ by the identification $\left.\begin{pmatrix} A & \\ & {}^tA^{-1}\end{pmatrix}\leftrightarrow A\right)$ which induce elations of the associated $(n-1)$-dimensional projective space with a same axis and distinct centers. Thus using Lemma 1 in [1], we can conclude that $H^x\cap L$ fixes a complete subspace of dimension $i=0$, 1, $n-3$ or $n-2$ of the $(n-1)$-dimensional projective space. Using the same argument repeatedly, we easily have the assertion of Lemma 2.

Lemma 3. *If q is odd, then H contains an element which is conjugate* (*in G*) *to the element u_1. If q is even, then H contains an element which is conjugate* (*in G*) *to either u_1 or u_2.*

Proof of Lemma 3. By Lemma 2, we may assume that $H^x\cap L$ contains a subgroup K defined in Lemma 2. Let Q_0 be the subgroup of Q defined by

$$Q_0=\left\{X\in GL(2n,q)\,;\,X=\begin{pmatrix} I_n & C\\ & I_n\end{pmatrix}\text{ with } C=\begin{array}{c} \begin{array}{ccc} \overbrace{}^{i} & & \overbrace{}^{2-i} \end{array}\\ \left(\begin{array}{c|c|c} 0 & 0 & 0\\ \hline 0 & D & 0\\ \hline 0 & 0 & 0\end{array}\right)\begin{array}{l} \}i\\ \\ \}2-i\end{array}\end{array},\ {}^tD=D\right\}.$$

Then $Q_0\cap H^x\neq\{1\}$, because $|G:H|\leqslant q^{2n}$ and n is sufficiently large. Now K normalizes Q_0, and we can conclude that, since $Q_0\cap H^x\neq\{1\}$, H^x contains an element $X=I_{2n}+e_{i+1,i+1}$ for q odd, and H^x contains an element

$X=I_{2n}+e_{i+1,i+1}$ or $X'=I_{2n}+e_{i+1,i+2}+e_{i+2,i+1}$ for q even. Thus the proof of Lemma 3 is completed since X is conjugate to u_1 and X' is conjugate to u_2.

Lemma 4. *If q is odd, then χ appears in $(1_{C_G(u_1)})^G$. If q is even, then χ appears in $(1_{C_G(u_1)})^G$ or $(1_{C_G(u_2)})^G$.*

Proof is the same as that of Lemma 3 in [2].

Lemma 5. *If q is odd, then χ appears in $(1_{G_{R-\{w_1\}}})^G$. If q is even, then χ appears in $(1_{G_{R-\{w_1\}}})^G$ or in $(1_{G_{R-\{w_2\}}})^G$.*

Proof of Lemma 5. By the theorem of Seitz [7], χ must appear in $(1_B)^G$, and by Lemma 4 χ must also appear in $(1_{C_G(u_1)})^G$ (or $(1_{C_G(u_2)})^G$ for q even). Thus we have the assertion of Lemma 5.

Lemma 6. *$(1_{G_{R-\{w_1\}}})^G$ is decomposed into 3 irreducible characters whose multiplicities are all 1. $(1_{G_{R-\{w_2\}}})^G$ is decomposed into 6 irreducible characters whose multiplicities are all 1 and three of which are contained in $(1_{G_{R-\{w_1\}}})^G$. Moreover for $q>2$ every irreducible character appearing in $(1_{G_{R-\{w_1\}}})^G$ or $(1_{G_{R-\{w_2\}}})^G$ are all of degree either not prime to q or $\leqslant q^{2n}-1$.*

Proof of Lemma 6. The irreducible characters appearing in $(1_{G_{R-\{w_1\}}})^G$ are of degrees 1, $\dfrac{q(q^n-1)(q^{n-1}+1)}{2(q-1)}$ and $\dfrac{q(q^{n-1}-1)(q^n+1)}{2(q-1)}$.

Now, the intersection matrix of the permutation group $(G, G/G_{R-\{w_2\}})$ is given as follows: rank is 6 and the subdegrees are $l_0=1$, $l_1=q(q+1)\cdot\dfrac{q^{2n-4}-1}{q-1}$, $l_2=(q+1)q^{2n-3}$, $l_3=\dfrac{q^4(q^{2n-4}-1)(q^{2n-6}-1)}{(q+1)(q-1)^2}$, $l_4=(q+1)q^{2n-2}\dfrac{q^{2n-4}-1}{q-1}$ and $l_5=q^{4n-5}$; the intersection matrix $M=(\mu_{ij}^{(1)})$ is given by

$$\left(\begin{matrix} 0 & 1 & 0 \\ q(q+1)\dfrac{q^{2n-4}-1}{q-1} & \dfrac{q^{2n-4}-1}{q-1}-2+q^2 & \dfrac{q^{2n-4}-1}{q-1} \\ 0 & q^{2n-4} & q^{2n-4}-1 \\ 0 & q^3\dfrac{q^{2n-6}-1}{q-1} & 0 \\ 0 & q^{2n-3} & q^2\dfrac{q^{2n-4}-1}{q-1} \\ 0 & 0 & 0 \end{matrix}\right. \qquad \text{(continued below)}$$

$$\left.\begin{matrix} 0 & 0 & 0 \\ (q+1)^2 & 1 & 0 \\ 0 & q & 0 \\ (q+1)\left(\dfrac{q^{2n-5}-q^2-q+1}{q-1}\right) & q\dfrac{q^{2n-6}-1}{q-1} & 0 \\ (q+1)^2q^{2n-5} & \dfrac{(2q^2-1)(q^{2n-5}-1)}{q-1} & (q+1)\dfrac{q^{2n-4}-1}{q-1} \\ 0 & q^{2n-3} & (q+1)(q^{2n-4}-1) \end{matrix}\right);$$

the eigenvalues of M are $\theta_0=q(q+1)\dfrac{q^{2n-4}-1}{q-1}$,

$\theta_1=\{q^{n-1}+(q^2+q-1)\}\left(\dfrac{q^{n-2}-1}{q-1}\right)$, $\theta_2=\{q^{n-1}-(q^2+q-1)\}\left(\dfrac{q^{n+2}+1}{q-1}\right)$,

$\theta_3=-(q+1)(q^{n-2}+1)$, $\theta_4=(q+1)(q^{n-2}-1)$ and $\theta_5=(q+1)(q^{n-2}-1)$.

We can easily evalute that the irreducible characters attached to the eigenvalues θ_3, θ_4 and θ_5 are of degree $>q^{2n}-1$ using Theorem 5.5 in D. G. Higman [6]. Moreover considering the irreducible characters of the Weyl group (cf. [3]), we can see that the irreducible characters attached to the eigenvalues θ_0, θ_1, θ_2 are those which appear in $(1_{G_{R-\{w_1\}}})^G$, and we have completed the proof of Lemma 6.

Thus we have completed the proof of Theorem 1 from Lemmas 1 $\sim$6 together with the well known facts that if $(q, |G:H|)=1$ then H is a parabolic subgroup and that $Sp(2n, q)$ has no parabolic subgroup of rank 2.

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